Symmetries in Physics:
Quick Guide

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Abstract

This is a quick guide – a summary – of the Symmetries in Physics course at the University of Manchester, taught by M.Dasgupta between Jan '09 and May '09. These summary notes are based upon his lecture notes. A copy of the full lecture notes, on this topic, may be found at `www.jpoffline.com`.

Keywords:

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I. GROUP THEORY

Given a set G and composition law $\circ$, we say that $(G, \circ)$ is a group if it obeys the **group axioms**,

- **Closure**; $\forall a, b \in G$, $a \circ b \in G$.
- **Identity**; $\exists e \in G$: $\forall a \in G$, $e \circ a = a \circ e = a$.
- **Inverse**; $\forall a \in G$, $\exists a^{-1} \in G$: $a \circ a^{-1} = a^{-1} \circ a = e$.
- **Associative**; $\forall a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.

An extra property that a group may have is commutativity. If $\forall a, b \in G$, $a \circ b = b \circ a$, then G is **Abelian**.

A. Properties & Classes of Groups

Order: the order of an element is the power to which that element must be raised to get the identity element. Finite groups have finite order.

Isomorphism: an isomorphism is a one-to-one relationship between elements of two different groups, $A \cong B$.

Subgroup: a subset H of a group G is a subgroup of G if the set is closed. A **proper subgroup** is one that is not just the identity or the entire group.

Conjugacy: we say that if $a = gbg^{-1}$, then a and b are conjugate elements, with conjugating element g.

Equivalence: a relation between elements is an equivalence relation if it satisfies

$$\text{reflexive}: \quad a \sim a; \quad \text{symmetry}: \quad a \sim b, b \sim a; \quad \text{transitive}: a \sim b, b \sim c, a \sim c.$$

Conjugacy is an example of an equivalence relation. An equivalence relation partitions a group into disconnected sets of elements that are equivalent; these are equivalence classes;

$$(a) = \{b | b = gag^{-1}; b, g \in G\}.$$

Abelian groups have each element in its own conjugacy class.

Coset: let H be a subgroup of G. Then, the left coset of g_i is

$$g_i H = \{g_i h_j\}, \quad g_i \in G, \forall h_j \in H.$$

Members of a coset are equivalent, and can hence be formed from any of its elements.

Normal subgroup: those subgroups H of G for which

$$gHg^{-1} = H \quad \Rightarrow \quad gh_i g^{-1} = h_j, \quad \forall g \in G, \forall h_i, h_j \in H.$$

Quotient group: for a normal subgroup H of G, the set of cosets $\{g_i H\}, \forall g_i \in G$, is the quotient group G/H. The quotient group has the multiplication law

$$(g_1 H) \circ (g_2 H) = (g_1 \circ g_2) H.$$

If $G \cong A \times B$ then $G/B \cong A$; but, the converse is not necessarily true.

Direct product of groups: a group G can be expressed as a direct product of its subgroups A and B if, and only if the elements of A and B commute, and every element in G can be uniquely expressed as the product of two elements in A and B;

$$a_i b_j = b_j a_i, \quad \forall a_i \in A, \forall b_j \in B; \qquad \forall g \in G, g = a_i b_j, a_i \in A, b_j \in B.$$

Hence, A and B are normal subgroups.

Homomorphism: a mapping $f : A \mapsto B$ between groups such that group multiplication is preserved. An element of B may be the image of more than one element of A; but, each element of A can be mapped into only one element of B. The group multiplication is

$$f(a_1 \circ a_2) = f(a_1) \star f(a_2).$$

Kernel: the elements in A that are mapped to the identity element of B, e_B, form the Kernel of the mapping,

$$\mathrm{Ker} f = \{a \in A | f(a) = e_B\}.$$

The Kernel $\mathrm{Ker} f$ is a normal subgroup of A.

B. Examples of Finite Groups

Some examples are

- The set of n integers, under addition modulo n, $(Z_n, +_n)$.
- The permutation group of n objects, $(S_n, \circ)$.
- The rotation of a regular n-gon, with directed sides, $(C_n, \circ)$.
- The dihedral group; symmetry operations upon an n-gon with undirected sides, $(D_n, \circ)$.

One can show that

$$C_n \cong Z_n, \quad C_6 \cong C_3 \times C_2, \quad C_6/C_3 \cong C_2$$

II. REPRESENTATIONS OF GROUPS

A representation, of $[n]$-dim of the abstract group G, is defined as a **homomorphism** $G \to GL(n, \mathbb{C})$; to the group of **non-singular** $n \times n$ matrices with complex elements. They must be non-singular due to the essential existence of the inverse; and **orthogonal** as length preserving: $R^{-1} = R^T$. That the mapping is a homomorphism means that group multiplication is preserved,

$$D(g_1)D(g_2) = D(g_1 g_2).$$

An example is the rotation of 3D vectors, about the z-axis; the matrix representation is

$$R(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can show

$$\psi'(x) = \psi(R^{-1}x).$$

If D is a rep, then so is SDS^{-1} (i.e. they are equivalent reps). To distinguish between equivalent reps, look at the **character**,

$$\chi = \{\chi(g) | g \in G\}, \quad \chi(g) = \text{Tr } D(g).$$

Equivalent reps have the same character χ.

A rep of $[n+m]$-dim is said to be reducible if $D(g)$ is of the form

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}.$$

Then, $A(g)$ is an $[m]$-dim rep, and $B(g)$ an $[n]$-dim rep of G. Hence, we write the direct sum,

$$D(g) = A(g) \oplus B(g).$$

The reps A and B could be reduced further, until we have the irreps of G.

A. The Vector Space

This is a space over the **field of complex numbers**, with two operations: **addition** and **multiplication**. The axioms defining the space are:

- **Addition**

  Closure: $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.

  Null vector: $\exists \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$, $\forall \mathbf{u} \in V$.

  Inverse: $\forall \mathbf{u} \in V$, $\exists (-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

  Associative: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

  Commutative: $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

  Hence, the vector space is an Abelian group under addition.

- **Multiplication**

  Closure: $\forall a \in \mathbb{C}, \mathbf{u} \in V$, then $a\mathbf{u} \in V$.

  Identity: $1 \in \mathbb{C}$ such that $\forall \mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$.

  Distributive: $\forall a, b \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, we have

  $$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$$

  Associative: $\forall a, b \in \mathbb{C}$, $\mathbf{u} \in V$, we have $a(b\mathbf{u}) = (ab)\mathbf{u}$.

A set of vectors $\{\mathbf{e}\}_{i=1}^m$ is **linearly independent** if there is no non-trivial combination which yields the null vector. That is, if $\lambda_i \mathbf{e}_i = \mathbf{0}$, the only solution is $\lambda_i = 0$, if the $\{\mathbf{e}\}_{i=1}^m$ are linearly independent.

A set of linearly independent vectors $\{\mathbf{e}\}_{i=1}^m$ forms a **basis** of V if they span the space; $\mathbf{u} = \lambda_i \mathbf{e}_i$.

The **dimension of a basis** is just the number of basis vectors.

Linear transformations are such that

$$\hat{T}\left(\alpha \mathbf{u} + \beta \mathbf{v}\right) = \alpha \hat{T}\mathbf{u} + \beta \hat{T}\mathbf{v}.$$

If we have

$$u'_j = D_{ji} u_i \quad \mathbf{e}'_i = \mathbf{e}_j S_{ji},$$

then the new matrix representing $\hat{T}$ in the new basis is

$$\hat{T}' = SDS^{-1}.$$

As an example, in the complex basis, we can write

$$R'(\beta) = \begin{pmatrix} e^{i\beta} & 0 & 0 \\ 0 & e^{-i\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that $R(\beta)$ and $R'(\beta)$ have equal trace, as they are equivalent – both rotations through β, but referred to different axes.

As the transformation is length preserving, we have $S^{-1} = S^\dagger$; i.e. S **is unitary**.

The **Clebsch-Gordan decomposition** is

$$D^{(\mu)} \otimes D^{(\nu)} = \sum_{\oplus\sigma} a_\sigma D^{(\sigma)},$$

where the $D^{(\sigma)}$ are the irreps. If there is no interaction in the composite system, we have $G \times G \to G \times G$; if there is interaction, then $G \times G \to G$.

III. CONTINUOUS GROUPS

A example is the $SO(N)$ group, which is the group of **special orthogonal rotation matrices in N-dim**. Hence, if $R \in SO(N)$, then $\det R = 1$ and $R^T = R^{-1}$.

A more concrete example is $R(\phi)$, the rotation about the z-axis in the $x-y$-plane;

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad R(\phi) \in SO(2).$$

This is the so-called **defining represetation** of a matrix in $SO(2)$. If we change to a complex basis, the same rotation can be expressed in the diagonal form

$$R(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

Hence, we say that the irreps of $SO(2)$ are $e^{-im\phi}$, where $m \in \mathbb{Z}$. The Clebsch-Gordan decomposition for composing rotations is hence just

$$SO(2): \quad D^{(m)} \otimes D^{(m')} = D^{(m+m')}.$$

A. Generators

We can always make a Taylor expansion of a matrix, for small argument,

$$R(\phi) \approx \mathbb{1} - iX\phi, \quad \phi \ll 1.$$

Hence, we have identified the **generator** of a group with rotation matrix R,

$$X = i\left.\frac{dR}{d\phi}\right|_{\phi=0}.$$

A property following the unitarity of R, $R^\dagger R = \mathbb{1}$, is that $X^\dagger = X$. Also, Tr $X = 0$. The rotation matrix is then

$$SO(2): \quad X = m \quad \Rightarrow \quad R(\phi) = e^{-iX\phi},$$

so that in $SO(2)$, the generator is just $X = m$. In $SO(3)$, the generator is

$$SO(3): \quad (X_a)_{jk} = -i\epsilon_{jka} \quad \Rightarrow \quad D(\phi) = e^{-i\hat{\mathbf{n}}\cdot\mathbf{X}\phi}.$$

In $SO(3)$, the generators do not commute,

$$[X_a, X_b] = i\epsilon_{abc}X_c,$$

a relationship which can be proved using $\epsilon_{ijk}\epsilon_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$.

In $SU(2)$ (where the matrices are now unitary, as well as having unit determinant), we have the generators and rotation matrices

$$SU(2): \quad X_i = \frac{1}{2}\sigma_i \quad \Rightarrow \quad D(\theta) = e^{-i\frac{1}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\theta},$$

where the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now, two distinct elements of $SU(2)$ map into the identity of $SO(3)$,

$$\text{Ker}\,\{SU(2) \longmapsto SO(3)\} \cong \mathbb{Z}_2,$$

hence by the isomorphism theorem,

$$SO(3) \cong SU(2)/\mathbb{Z}_2.$$

B. SU(N) Tensors

Particles are represented by spinors $\psi'_a = U_{ab}\psi_b$, and anti-particles in the conjugate representation, $\psi'^a = U^*_{ab}\psi^b$.

In $SU(2)$, $U^* = CUC^{-1}$ where $C = i\sigma_2$. Hence, in $SU(2)$ the two transformations are equivalent.

We have that C raises indices,

$$\psi^a = C_{ab}\psi_b = \epsilon_{ab}\psi_b,$$

where one can show that C is invariant,

$$C = U^T C U.$$

In $SU(2)$, **scalar invariants** are formed via

$$\epsilon^{ab}\psi_a\phi_b = \psi^a\phi_a = \psi'^a\phi'_a$$

We symmetrise and anti-symmetrise indices according to

$$\psi_{(a}\phi_{b)} = \frac{1}{2}\left(\psi_a\phi_b + \psi_b\phi_a\right), \quad \psi_{[a}\phi_{b]} = \frac{1}{2}\left(\psi_a\phi_b - \psi_b\phi_a\right).$$

In $SU(3)$, a scalar is

$$\psi_a\phi^a = \epsilon^{abc}\psi_a\chi_{[bc]}.$$

C. Young Tableaux

A diagram that conforms to the rules:

- For $SU(N)$, the number of rows is $\leq N$.
- Each row has no more boxes than the one above it.
- Represents a tensor after process of symmetrisation and anti-symmetrisation.

1. *Dimensionality*

Compute dimension of a diagram via

- $\mathcal{N}$ = product of entries, where one starts at the top left with N, incrementing across, decrementing down.
- $\mathcal{D}$ = products of entries, where each entry is the number of boxes to the right plus the number below, plus one for itself.
- Dimension $= \mathcal{N}/\mathcal{D}$.

2. *C-G Decomposition*

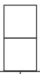
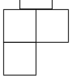
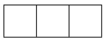
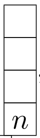
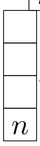
To decompose a composite system into the irreps, one should follow the set of rules:

- Write down tableaux.
- Label successive rows of the second diagram (every box in a given row has the same label).
- Add boxes from the second diagram to the first, such that
  - the agumented diagram is always legal,
  - boxes with the same label must not appear in the same column,
  - at each box, $n_a \geq n_b \geq n_c$, where n_i is the number of boxes above and to the right with label i.

- Remove multiple identical diagrams.
- Excise columns with N rows from diagrams in $SU(N)$.

If there is no scalar in the decomposition, then the composite system is unable to be observed in nature.

3. *Examples of Dimensionality*

	Dimension of diagram in			
diagram	$SU(2)$	$SU(3)$	$SU(6)$	$SU(N)$
	1	3	6	N
	1	$\bar{3}$	10	$\frac{1}{2}N(N-1)$
	2	8	70	$\frac{1}{3}N(N+1)(N-1)$
	4	10	56	$\frac{1}{6}N(N+1)(N+2)$
 $, n = N$	1	1	1	1
 $, n = N-1$	1	$\bar{3}$	$\bar{6}$	$\bar{N}$

D. Pieces of Lie Algebra

The generators form a group under addition and commutation,

$$[X_\alpha, X_\beta] = i f_{\alpha\beta\gamma} X_\gamma,$$

where $f_{\alpha\beta\gamma}$ are the **structure constants** (note: which are zero in an Abelian group), and form the **adjoint representation**,

$$(T_\alpha)_{\beta\gamma} = -i f_{\alpha\beta\gamma}.$$

The **Casimir operator** is

$$T^2 = \sum_\alpha T_\alpha T^\alpha \propto I,$$

where the last part is **Schur's lemma**. The Casimir operator commutes with all the generators

$$\left[T^2, T_\alpha\right] = 0, \quad \forall\alpha.$$

An example of a Casimir operator is the angular momentum operator, J^2 from quantum mechanics.

IV. SYMMETRIES OF THE LAGRANGIAN

The simplest example is from Maxwell's equations. We introduce the potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

and if we change the potentials via the gauge (i.e. position dependent) transformations

$$\mathbf{A} \longmapsto \mathbf{A} + \nabla\Omega, \quad \phi \longmapsto \phi - \frac{\partial \Omega}{\partial t},$$

the observable fields are unchanged. In covariant notation, we make the definitions

$$A^\mu = (\phi/c, \mathbf{A}), \quad J^\mu = (\rho c, \mathbf{J}), \quad \partial^\mu = (\tfrac{1}{c}\partial_t, -\nabla).$$

Hence, the Lorentz gauge reads $\partial^\mu A_\mu = 0$. The EM **field strength tensor** is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

To make the non-relativistic Schrodinger equation gauge invariant, we make the **minimal substitution** of the momentum operator,

$$\hat{\mathbf{p}} \longmapsto \hat{\mathbf{p}} - e\mathbf{A},$$

that is, we change the derivative operator

$$-i\hbar\nabla \longmapsto -i\hbar\nabla - e\mathbf{A},$$

where

$$\mathbf{A} \longmapsto \mathbf{A}' = \mathbf{A} + \frac{\hbar}{e}\nabla\alpha.$$

Then, the non-relativistic Hamiltonian

$$H = \frac{\hat{\mathbf{p}}^2}{2m} + V$$

is gauge invariant. The point of this is, that we had to introduce a **gauge field**, with its transformation rule.

A. Lagrangians for Various Fields

The **Klein-Gordon Lagrangian** is

$$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2,$$

which gives the equation of motion describing **scalar bosons**

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0.$$

The **Dirac Lagrangian** is

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi,$$

which gives the equation of motion describing free fermions

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0.$$

The **Maxwell Lagrangian** is

$$\mathcal{L}_{\mathrm{M}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{(\partial^\mu A_\mu)^2}{2\xi},$$

which gives the equation of motion describing free photons

$$\partial^\mu F_{\mu\nu} = 0.$$

The ξ-term is a **gauge-fixing** term, and has no physical consequences.

B. Noether's Theorem

If we have the rotation matrix $U = e^{i\theta_a T^a}$ for a Lie group, where the T_a are the generators, then small changes in the field can be expanded to find

$$\delta\phi_i = i\theta_a \left(T^a\right)_{ij}\phi_j, \quad \theta_a \ll 1.$$

If we **require an invariant Lagrangian** under this transformation, we find

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} i\theta_a (T^a)_{ij}\phi_j\right) = 0.$$

If the rotation parameters θ_a are global, then this reduces to a **continuity equation**

$$\partial_\mu J^\mu = 0, \quad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} i(T^a)_{ij}\phi_j.$$

So, writing $J^\mu = (\rho, \mathbf{J})$, we also have a **conserved charge**

$$Q = \int \rho \, dV, \quad \frac{dQ}{dt} = 0.$$

Therefore, **requiring an invariant Lagrangian under global transformation gives a conserved 4-current and charge**. This is Noether's theorem.

C. Abelian Gauge Symmetry

If the symmetry group doing the gauge transforming is Abelian, we introduce a gauge field A_μ and covariant derivative $\mathcal{D}_\mu$; where

$$\begin{aligned} A_\mu \longmapsto A'_\mu &= A_\mu + \partial_\mu \Lambda, \\ \mathcal{D}_\mu &= \partial_\mu - ieA_\mu. \end{aligned}$$

In which case, the Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left(i\gamma^\mu \mathcal{D}_\mu - m\right)\psi$$

is gauge invariant (i.e. invariant under position dependent transformations). In particular, we have an invariant kinetic term,

$$\bar{\psi}' \mathcal{D}'_\mu \psi' = \bar{\psi} \mathcal{D}_\mu \psi.$$

By introducing the gauge field, we induce an interaction between the fermions and the gauge field,

$$j^\mu = e\bar{\psi}\gamma^\mu\psi.$$

An example of such a group is the $U(1)$ group (in which case the gauge-fermion interaction is photon-fermion). The field strength tensor is unchanged.

D. Non-Abelian Gauge symmetry

If the symmetry group doing the gauge transforming is non-Abelian, we introduce a gauge field A_μ and covariant derivative $\mathcal{D}_\mu$; where

$$\begin{aligned} A_\mu \longmapsto A'_\mu &= U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \\ \mathcal{D}_\mu &= \partial_\mu - igA_\mu . \end{aligned}$$

These rules make the kinetic term $\bar{\psi}\mathcal{D}_\mu\psi$ gauge invariant.

The non-Abelian terms require us to change the field-strength tensor to

$$F_{\mu\nu} = \mathcal{D}_\mu A_\nu - \mathcal{D}_\nu A_\mu .$$

Then, the full gauge-invariant Lagrangian reads

$$\mathcal{L}_{\text{n-A}} = \bar{\psi}\left(i\gamma^\mu \mathcal{D}_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

The new definition of $F_{\mu\nu}$ gives rise to self-interaction between the gauge fields, as well as the fermion-gauge interaction from the kinetic term.

If the group is $SU(2)$, the theory describes the weak interaction, and the gauge fields are the three gauge bosons $W^\pm, Z^0$.

If the group is $SU(3)$, the theory describes QCD, and the gauge field are the 8 gluons.