Relativistic Quantum Mechanics: Quick Guide

J.Pearson*

(Dated: May 30, 2009)

Abstract

This is a quick guide – a summary – of the Relativistic Quantum Mechanics course at the University of Manchester, taught by G.Shaw between Jan '09 and May '09. These summary notes are based upon his lecture notes. A copy of the full lecture notes, on this topic, may be found at `www.jpoffline.com`.

Keywords:

*Electronic address: jon@jpoffline.com

Contents

I. REVIEW OF NON-RELATIVISTIC QUANTUM MECHANICS

We use the **flat Minkowski** metric, with signature

$$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)\,,$$

so that the D'Alembertian is

$$\Box = \partial_\mu \partial^\mu = \left(\frac{\partial^2}{\partial t^2}, -\nabla^2\right).$$

The **equation of continuity** is

$$\partial_\mu j^\mu = 0, \quad j^\mu = (\rho, \mathbf{j});$$

which obviously expands out of the summation convention to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

Associated with the charge density, ρ, is a conserved (integrated) charge,

$$Q = \int d^3x\, \rho \quad \Rightarrow \quad \frac{dQ}{dt} = 0.$$

The **non-relativistic Schrodinger equation** (SE) is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi.$$

To compute ρ and $\mathbf{j}$, we take the conjugate of this, and multiply across by ψ^*, and vice-versa then subtract. To satisfy the continuity equation, one finds that

$$\rho = |\psi|^2, \quad \mathbf{j} = \frac{i\hbar}{2m}\left(\psi \nabla \psi^* - \psi^* \nabla \psi\right).$$

Notice that $\rho \geq 0$, so that ρ can be interpreted as a probability density for position – the **Born interpretation** of quantum mechanics.

From hereon, we set $\hbar = c = 1$.

The free SE can be modified to include interactions with a potential; to do so, use the **minimal substitution**,

$$\partial_\mu \longmapsto \partial_\mu + iqA_\mu, \quad A_\mu = (\phi, -\mathbf{A}).$$

II. THE KLEIN-GORDON EQUATION

We use the **relativistic Hamiltonian**,

$$H = \sqrt{-\nabla^2 + m^2},$$

with relativistic SE,

$$H\phi = i\frac{\partial \phi}{\partial t}, \quad \phi = \phi(t, \mathbf{x}).$$

We avoid the square-root in the Hamiltonian by squaring, so that the SE becomes

$$H^2\phi = -\frac{\partial^2 \phi}{\partial t^2}.$$

Hence, we arrive at the **Klein-Gordon equation**,

$$\left(\Box + m^2\right)\phi = 0.$$

Through the same method as before, we can compute the **charge** and **current**,

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi\frac{\partial \phi^*}{\partial t}\right), \quad \mathbf{j} = -i\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right).$$

This time, the charge density ρ is not positive-definite, and is hence **not a probability density**.

Inserting a **plane wave ansatz**, $\phi(x) = e^{-ipx}$, where $px = p_\mu x^\mu = Et - \mathbf{p}\cdot\mathbf{x}$, we compute that

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad E = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

A. The "Hydrogen Atom"

Using the **minimal substitution**, $\partial_\mu \mapsto \partial_\mu + iqA_\mu$, the KG equation becomes

$$\left[\left(\partial_\mu + iqA_\mu\right)\left(\partial^\mu + iqA^\mu\right) + m^2\right]\phi = 0.$$

In the hydrogen atom model, we use a potential $A_\mu = (qV(\mathbf{x}), 0)$ with ansatz

$$\phi(t, \mathbf{x}) = \psi(\mathbf{x})e^{-iEt},$$

so that the KG equation becomes

$$\left[-(E - V)^2 - \nabla^2 + m^2\right]\psi = 0.$$

We can then compare with the non-relativistic SE's result, to find the KG energy levels

$$E = m\left(1 + \frac{(Z\alpha)^2}{n'^2}\right)^{-1/2},$$

where $n' = n_{\mathrm{r}} + \ell' + 1$ and the relation of ℓ' to the ℓ of the non-relativistic SE is

$$\ell' = -\frac{1}{2} \pm \sqrt{\left(\ell + \frac{1}{2}\right)^2 - (Z\alpha)^2}.$$

a. The Klein Paradox Considering a step-function potential, where we solve the KG equation either side of the step; equating the wavefunction and derivatives to find the constants of integration. We then compute the current and find that the wave and currents direction of travel are different – this is "explained" by introducing particle/anti-particle creation at the barrier.

III. THE DIRAC EQUATION

The proposal of Hamiltonian is

$$\begin{aligned} H &= -i\boldsymbol{\alpha}\cdot\nabla + \beta m \\ &= \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m. \end{aligned}$$

Hence, the SE becomes the **Dirac equation**:

$$(-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\,\psi = i\frac{\partial\psi}{\partial t}.$$

This Hamiltonian satisfies $E^2 = \mathbf{p}^2 + m^2$ if

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\beta, \alpha_i\} = 0, \quad \beta^2 = 1.$$

Following these properties, one can deduce that

$$\text{Tr }\alpha_i = \text{Tr }\beta = 0.$$

The **Dirac representation** of these α_i, β matrices is

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix};$$

where the **Pauli matrices** are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices can be shown to satisfy

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c, \quad \{\sigma_a, \sigma_b\} = 2\delta_{ab}.$$

The proposal is that ψ is a 4-component wave-equation. Notice that the Hermitian conjugate of the Dirac equation is

$$i\nabla\cdot\left(\psi^\dagger\boldsymbol{\alpha}\right) + m\psi^\dagger\beta = -i\frac{\partial\psi^\dagger}{\partial t}.$$

Hence, we compute the **charge** and **current**,

$$\rho = \psi^\dagger\psi, \quad \mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi.$$

A. Angular Momentum & Spin

The **orbital angular momentum** operator is

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} \quad \Rightarrow \quad L_i = \epsilon_{ijk} x_j p_k.$$

If an operator A is conserved, then $[H, A] = 0$. We find that

$$[H, \mathbf{L}] = -i\boldsymbol{\alpha} \times \mathbf{p} \quad \Rightarrow \quad [H, L_a] = -i\epsilon_{abc}\alpha_b p_c.$$

In light of this **non-conservation of orbital angular momentum**, we propose a new operator

$$\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix},$$

so that

$$[\alpha_a, \Sigma_b] = 2i\epsilon_{abc}\alpha_c, \quad [H, \Sigma_a] = 2i\epsilon_{abc}\alpha_b p_c.$$

Therefore, we define the **total angular momentum**,

$$\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma},$$

whereby

$$[H, J_i] = 0, \quad \forall i.$$

We define a **spin operator** $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$, which has eigenvalues $s = \pm\frac{1}{2}$.

B. Plane Wave States

A useful, easily derivable relation is

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2.$$

We find **positive energy states**

$$\psi^{(+)}_{\mathbf{p},s} = e^{-ipx}\sqrt{\frac{E+m}{2EV}}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi_s \end{pmatrix},$$

and **negative energy states**

$$\psi^{(-)}_{\mathbf{p},s} = e^{ipx}\sqrt{\frac{E+m}{2EV}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi_{-s} \\ \chi_{-s} \end{pmatrix};$$

where

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The **Dirac hole theory** explains away the presence of a negative energy by introducing a full sea of negative-energy states; so that upon a transition of a particle from the negative to positive energy states, it appears as an anti-particle.

C. Covariance of the Dirac Equation

We introduce the **γ-matrices**,

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i \quad \Rightarrow \quad \gamma^\mu = (\beta, \beta\alpha^i).$$

Hence, the Dirac equation can be written as

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0.$$

The γ-matrices can be easily shown to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0.$$

We introduce the **Dirac adjoint**,

$$\bar{\psi} = \psi^\dagger \gamma^0,$$

so that the 4-current can be written

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

1. *Transformations*

We transform the coordinate and wavefunction according to

$$x \longmapsto x' = ax, \quad \psi(x) \longmapsto \psi'(x') = S(a)\psi(x).$$

Then, the Dirac equation and j^μ are covariant, provided

$$S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu, \quad S^{-1} = \gamma^0 S^\dagger \gamma^0.$$

Lorentz boosts have

$$S(a) = e^{\omega\alpha_1/2} = \mathbb{1}\cosh\tfrac{\omega}{2} + \alpha_1 \sinh\tfrac{\omega}{2},$$

where

$$\sinh\omega = \gamma v, \quad \cosh\omega = \gamma; \quad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

The **parity** operation is

$$t \longmapsto t' = t, \quad x_i \longmapsto x_i' = -x_i.$$

Hence,

$$S(a) = P = \eta\gamma^0, \quad \eta = \pm 1.$$

By convention, we choose $\eta = 1$. We find that

$$P\psi^{(+)}_{\mathbf{p},s} = \eta\psi^{(+)}_{-\mathbf{p},s}, \quad P\psi^{(-)}_{\mathbf{p},s} = -\eta\psi^{(-)}_{-\mathbf{p},s},$$

so that particles and anti-particles have **opposite intrinsic parity**.

D. Interactions with Fields

Using the minimal substitutions

$$\partial_\mu \longmapsto \partial_\mu + iqA_\mu, \quad m \longmapsto m + S,$$

we can derive that for electrons in a **homogeneous magnetic field** (under the Coulomb gauge),

$$\left(-\frac{1}{2m}\nabla^2 + qA^0 - \boldsymbol{\mu}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}^2\right)\phi = \epsilon\phi,$$

where

$$\boldsymbol{\mu} = \frac{q}{2m}\left(\mathbf{L} + 2\mathbf{S}\right),$$

which is the prediction of the **gyromagnetic ratio** $g_\mathrm{s} = 2$ for electrons.

Assuming a **spherical potential** $V(r) = qA^0(r)$ (i.e. with $\mathbf{A} = 0$), and wavefunction ansatz

$$\psi(r) = \begin{pmatrix} f(r)\chi_s \\ g(r)i\sigma\cdot\hat{\mathbf{r}}\chi_s \end{pmatrix},$$

we expand the Dirac equation to find

$$\begin{aligned} \left(\frac{d}{dr} + \frac{2}{r}\right)g(r) + (m + S + V - E)\,f(r) &= 0, \\ \frac{df(r)}{dr} + (m + S - V + E)\,g(r) &= 0. \end{aligned}$$

These equations can then be used in the **MIT bag model**, and in a **hydrogen atom model**. The energy levels of the hydrogen atom are correct, up to the Lamb shift, and a very small correction to g_s.

IV. QUANTUM FIELDS

We use the following notation

$$E_{\mathbf{p}} = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}, \quad px = p^\mu x_\mu = Et - \mathbf{p}\cdot\mathbf{x}.$$

The general method for quantising a field is to first expand in terms of plane wave modes, with some coefficients which conform to a given commutation relation. Following this specification, we then substitute our field expansion (which is now an operator) into quantities, such as H or Q, to get a new operator. This operator then acts upon states. We will have

$$a_n^\dagger(\mathbf{p})\,|0\rangle = |\mathbf{p}\rangle\,.$$

A. Bosonic Fields: KG

The **Klein-Gordon field operator** is

$$\phi(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}V}} \left(a(\mathbf{p})e^{-ipx} + c^\dagger(\mathbf{p})e^{ipx}\right),$$

where we have that ϕ is generally non-Hermitian. We impose the **commutation relations**

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{p}')\right] = \delta_{\mathbf{p},\mathbf{p}'} = \left[c(\mathbf{p}), c^\dagger(\mathbf{p}')\right].$$

These relations make states symmetric. The **Hamiltonian** is

$$\begin{aligned} H \;&=\; \int d^3x \left[\frac{\partial\phi^\dagger}{\partial t}\frac{\partial\phi}{\partial t} + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi\right] \\ &=\; \sum_{\mathbf{p}} E_{\mathbf{p}}\left(a^\dagger(\mathbf{p})a(\mathbf{p}) + c^\dagger(\mathbf{p})c(\mathbf{p}) + 1\right), \end{aligned}$$

or, the **time-ordered** (i.e. ignore the infinite vacuum energy by measuring relative to the vacuum),

$$:H: = \sum_{\mathbf{p}} E_{\mathbf{p}}\left(a^\dagger(\mathbf{p})a(\mathbf{p}) + c^\dagger(\mathbf{p})c(\mathbf{p})\right).$$

The **Heisenberg equation of motion** holds with this field expansion, set of commutators and Hamiltonian,

$$i\frac{\partial\phi}{\partial t} = [\phi, H].$$

We can then compute

$$[H, a(\mathbf{p})] = -E(\mathbf{p})a(\mathbf{p}), \quad [H, a^\dagger(\mathbf{p})] = E(\mathbf{p})a^\dagger(\mathbf{p}).$$

The conserved **4-current density** is

$$j^\mu = iq\left(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi\right),$$

and related **conserved charge**,

$$\begin{aligned} Q &= \int d^3x\, j^0 \\ &= q\sum_{\mathbf{p}}\left(a^\dagger(\mathbf{p})a(\mathbf{p}) - c^\dagger(\mathbf{p})c(\mathbf{p})\right); \end{aligned}$$

where we have written the time-ordered charge. Hence, we can note that **Hermitian fields are not charged**.

We have the following interpretations for the coefficients:

- a: decreases charge & energy: **destroys particles**,
- $a^\dagger$: increases charge & energy: **creates particles**,
- c: increases charge & decreases energy: **destroys anti-particles**,
- $c^\dagger$: decreases charge & increases energy: **creates anti-particles**.

B. Fermionic Fields: Dirac

The **Dirac field operator** is

$$\psi(x) = \sum_{s,\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}V}}\left(b_s(\mathbf{p})u_s(\mathbf{p})e^{-ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ipx}\right),$$

where we impose the **anti-commutation relations**,

$$\left\{b_s(\mathbf{p}), b_s^\dagger(\mathbf{p}')\right\} = \delta_{\mathbf{p},\mathbf{p}'}\delta_{ss'} = \left\{d_s(\mathbf{p}), d_s^\dagger(\mathbf{p}')\right\}.$$

Hence, such states produced are **anti-symmetric**. The **spinors are orthonormal**,

$$u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = 2E_{\mathbf{p}}\delta_{ss'} = v_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p}).$$

We have the time-ordered **Hamiltonian**

$$\begin{aligned} H &= \int d^3x\, \psi^\dagger\left(-i\boldsymbol{\alpha}\cdot\nabla + \beta m\right)\psi \\ &= \sum_{s,\mathbf{p}} E_{\mathbf{p}}\left(b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p})d_s(\mathbf{p})\right). \end{aligned}$$

The **conserved charge** is

$$\begin{aligned} Q &= q\int d^3x\,\psi^\dagger\psi \\ &= q\sum_{s,\mathbf{p}}\left(b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p})d_s(\mathbf{p})\right). \end{aligned}$$

We can then compute

$$[H, b_s(\mathbf{p})] = -E(\mathbf{p})b_s(\mathbf{p}), \quad [H, b_s^\dagger(\mathbf{p})] = E(\mathbf{p})b_s^\dagger(\mathbf{p}).$$

Hence, we make the same interpretation of b/d as a/c.

C. The Feynman Propagator

We define

$$G_\mathrm{F}(x) = -i\,\langle 0|\,T\left\{\phi(x)\phi^\dagger(0)\right\}|0\rangle$$

to be the **Feynman propagator**. If $t > 0$, then this describes the **creation** of a particle at $t = 0$, and its **destruction** at $t = t$. Conversely, if $t < 0$, an anti-particle is created at $t = t$ and destroyed at $t = 0$. This can be cast into the form

$$G_\mathrm{F}(x) = -i\sum_{\mathbf{p}}\left(\theta(t)\frac{e^{-ipx}}{2E_\mathbf{p}V} + \theta(-t)\frac{e^{ipx}}{2E_\mathbf{p}V}\right),$$

where

$$\theta(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases}$$

One can show that G_F is a Green function for the KG equation,

$$\left(\partial^\mu\partial_\mu + m^2\right)G_\mathrm{F} = -\delta^4(x).$$

Hence, using contour integration, one can show

$$iG_\mathrm{F}(x - x') = \int\frac{d^4q}{(2\pi)^4}e^{-iq(x-x')}\frac{i}{q^2 - m^2 + i\epsilon},$$

where the $(+i\epsilon)$-term is due to the **Feynman prescription** and m is the mass of the particle that is created/destroyed by G_F.

D. The Interaction Picture

In the **Schrodinger picture**, states evolve and operators are constant. Conversely, in the **Heisenberg picture**, the states are constant and operators evolve. We have that

$$|\psi(t)\rangle_{\mathrm{S}} = e^{-iHt}|\psi(0)\rangle_{\mathrm{S}}, \quad |\psi\rangle_{\mathrm{H}} = |\psi(0)\rangle_{\mathrm{S}}, \quad A_{\mathrm{H}}(t) = e^{iHt}A_{\mathrm{S}}e^{-iHt}.$$

In writing these, the expectation values are unchanged. We have the **Heisenberg equation of motion** which holds for interaction picture operators

$$-i\frac{dA_{\mathrm{I}}}{dt} = [H, A_{\mathrm{I}}(t)],$$

where

$$H = H_0 + H_{\mathrm{I}},$$

and H_0 is the free Hamiltonian.

E. The S-matrix

States evolve according to

$$|\psi(t)\rangle_{\mathrm{I}} = U(t, t_0)|\psi(0)\rangle_{\mathrm{I}},$$

where

$$i\frac{d}{dt}U(t, t_0) = H_{\mathrm{I}}U(t, t_0).$$

Thus, we define

$$S = U(\infty, -\infty),$$

so that

$$|\psi(\infty)\rangle_{\mathrm{I}} = |\psi_{\mathrm{f}}\rangle_{\mathrm{I}} = \sum_i S_{\mathrm{fi}}|\psi_{\mathrm{i}}\rangle_{\mathrm{I}}.$$

Hence, the amplitude of a process is

$$S_{\mathrm{fi}} = \langle\psi_{\mathrm{f}}|\, S \,|\psi_{\mathrm{i}}\rangle\,.$$

The S-matrix is **unitary**; having the consequence of normalisation conservation.

The time-ordered matrix to n^{th} order is

$$S^{(n)} = \frac{(-i)^n}{n!}\int d^4x_1 \ldots \int d^4x_n T\left\{\mathcal{H}(t_1)\ldots\mathcal{H}(t_n)\right\}.$$

V. QUANTISED PROCESSES

Before we start anything, let us present some useful results:

$$\begin{aligned}
\delta^{(4)}(x-x') &= \delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}'),\\
\int d^4x\, e^{-ix(k-p)} &= (2\pi)^4\delta^{(4)}(k-p),\\
VT\delta_{kp} &= (2\pi)^4\delta^{(4)}(k-p),\\
\frac{1}{V}\sum_{\mathbf{k}} &\longmapsto \int \frac{d^3k}{(2\pi)^3},\\
\delta(f(x)) &= \sum_{x_0}\frac{\delta(x-x_0)}{|f'(x_0)|}, \quad f(x_0)=0.
\end{aligned}$$

A. Decays

We shall illustrate the main points by considering a ϕ^3-decay theory. Suppose we have a scalar boson decay to a fermion and anti-fermion,

$$\mathbf{p} \longrightarrow (\mathbf{k}, s) + (\mathbf{k}', s').$$

The interaction Hamiltonian for this process is

$$\mathcal{H}_{\mathrm{I}} = gN\left(\bar{\psi}(x)\psi(x)\phi(x)\right),$$

where ϕ is a neutral bosonic field, and ψ a charged Dirac (i.e. fermionic) field. The only term to contribute to the S-matrix integrand, after substituting in the field operator expansions, is

$$\langle \mathrm{f}|\, \bar{\psi}^{(-)}(x)\psi^{(-)}(x)\phi^{(+)}(x)\, |\mathrm{i}\rangle .$$

Reading from left to right: **create** fermion, **create** anti-fermion, **destroy** boson. As the initial state has a single particle of momentum $\mathbf{p}$, we must create that state from the vacuum, so we write

$$\begin{aligned}
\phi^{(+)}\,|\mathrm{i}\rangle &= \phi^{(+)}a^\dagger(\mathbf{p})\,|0\rangle\\
&= \sum_{\mathbf{p}'}\frac{1}{(2V\omega_{\mathbf{p}'})}a(\mathbf{p}')a^\dagger(\mathbf{p})e^{-ipx}\,|0\rangle .
\end{aligned}$$

Then, after using the standard commutation relation, we have

$$\phi^{(+)}\,|\mathrm{i}\rangle = \frac{e^{-ipx}}{(2V\omega_{\mathbf{p}'})}\,|0\rangle .$$

In an entirely analogous manner, we have

$$\psi^{(+)} \left|\mathrm{f}\right\rangle = \psi^{(+)} b_s^\dagger(\mathbf{k}) \left|0\right\rangle = \frac{e^{-ikx}}{(2VE_{\mathbf{k}'})^{1/2}} u_s(\mathbf{k}) \left|0\right\rangle .$$

The adjoint of this appears in the matrix element, $\left\langle \mathrm{f}\right| \bar{\psi}^{(-)}$, so that we pick up a $\bar{u}_s(\mathbf{k})$. Putting everything together, one gets

$$S_{\mathrm{fi}} = (\mathrm{norm}) \times (2\pi)^4 \delta^{(4)}(k + k' - p) \times \mathcal{M}_{\mathrm{fi}},$$

where the normalisation term contains all factors of energy and $2V$; the dynamics are contained in the **Feynman amplitude** $\mathcal{M}_{\mathrm{fi}}$ – later on we will give the assignment of factors to a Feynman diagram. For this process, we have the Feynman amplitude

$$\mathcal{M}_{\mathrm{fi}} = -ig\bar{u}_s(\mathbf{k}) v_{s'}(\mathbf{k}').$$

1. Decay Rates

The probability of decay, per unit time is

$$\Gamma = \sum_{\mathrm{f}} \frac{|S_{\mathrm{fi}}|^2}{T}.$$

In the **rest frame of the decaying particle** we have

$$p_{\mathrm{i}}^\mu = (m_{\mathrm{i}}, \mathbf{0}), \quad \mathbf{k}_{\mathrm{f}} = -\mathbf{k}_{\mathrm{f}}', \quad E(\mathbf{k}_{\mathrm{f}}) = E(\mathbf{k}_{\mathrm{f}}') = \frac{m_{\mathrm{i}}}{2}.$$

Hence, in this frame, we have

$$k^\mu k'_\mu = E^2(\mathbf{k}) - \mathbf{k} \cdot \mathbf{k}' \ = \ E^2(\mathbf{k}) + \mathbf{k}^2, \quad \mathbf{k}^2 = E^2(\mathbf{k}) - m_{\mathrm{f}}^2.$$

Combining, one easily derives

$$k^\mu k'_\mu = \frac{m_{\mathrm{i}}^2}{4} - m_{\mathrm{f}}^2.$$

This result is useful when evaluating the spin-sums.

If computing **quark decay rates**, the result must be **multiplied by 3** (relative to the lepton decay rate), as three colour states in the phase space into which quarks can decay.

B. Scattering

To describe the scattering of two particles, one needs the second order matrix element $S_{\text{fi}}^{(2)}$. When computing the integrand of the S-matrix, one finds a term

$$\langle 0|\, T\left(\phi(x)\phi(x')\right)|0\rangle = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')} \frac{i}{q^2 - m^2 + i\epsilon}.$$

The equality follows from our discussion on the propagator. We have that q is the momentum transfered by the exchange boson, which has mass m.

A typical Feynman amplitude is

$$\mathcal{M}_{\text{fi}} = (-ig_1)\bar{u}_{1,s}(\mathbf{p})u_{1,s'}(\mathbf{p}')\frac{i}{q^2 - m^2 + i\epsilon}(-ig_2)\bar{u}_{2,t}(\mathbf{k})u_{2,t'}(\mathbf{k}'),$$

which corresponds to the process

$$(\mathbf{p}, s) + (\mathbf{k}, t) \longrightarrow (\mathbf{p}', s') + (\mathbf{k}', t').$$

If we have identical particles scattering, then we have two contributions,

$$\mathcal{M}_{\text{fi}} = \mathcal{M}_{\text{fi}}^{\text{direct}} + \mathcal{M}_{\text{fi}}^{\text{exchange}},$$

where

$$\mathcal{M}_{\text{fi}}^{\text{exchange}}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4) = \pm\mathcal{M}_{\text{fi}}^{\text{direct}}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_4\mathbf{p}_3),$$

with $+$ for bosons, and $-$ for fermions.

C. Cross-sections

A cross-section is the thing we actually measure with a detector.

The **transition rate** is

$$w_{\text{fi}} = \frac{|S_{\text{fi}}|^2}{T}.$$

The **flux** is

$$f = \frac{1}{V}|\mathbf{v}_1 - \mathbf{v}_2|,$$

where V is some volume, and the $\mathbf{v}_i$ are the velocities of the incoming particles.

The **final phase space** is

$$\frac{V}{(2\pi)^3}d^3p_1' \frac{V}{(2\pi)^3}d^3p_2'.$$

The **cross-section** σ is defined to be the transition rate into a given set of final states, per unit flux of initial particles. The units are $[\sigma] = L^2$. Hence,

$$d\sigma = \frac{V}{|\mathbf{v}_1 - \mathbf{v}_2|} \frac{|S_{\rm fi}|^2}{T} \frac{V}{(2\pi)^3} d^3p'_1 \frac{V}{(2\pi)^3} d^3p'_2,$$

which can be written as

$$d\sigma = \frac{1}{F} |\mathcal{M}_{\rm fi}|^2 dQ,$$

where

$$\begin{aligned} F &= 4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|, \\ dQ &= (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2) \frac{d^3p'_1}{(2\pi)^3 2E'_1} \frac{d^3p'_2}{(2\pi)^3 2E'_2}. \end{aligned}$$

VI. NON-QED/QCD FEYNMAN RULES

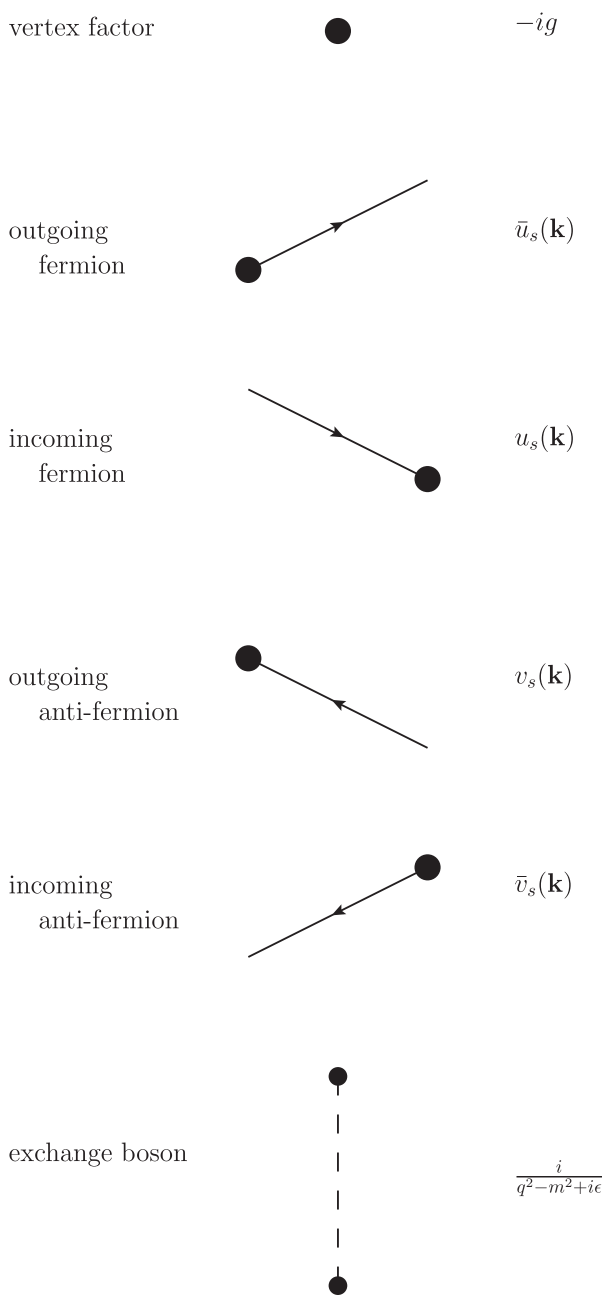

FIG. 1: The Feynman rules. The arrows do not denote direction of travel.