Physics of Fluids:
Quick Guide

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Abstract

This is a quick guide – a summary – of the Physics of Fluids course at the University of Manchester, taught by T.Mullin between Jan '09 and May '09. These summary notes are based upon his lecture notes. A copy of the full lecture notes, on this topic, may be found at `www.jpoffline.com`.

Keywords:

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I. INTRODUCTORY DEFINITIONS

A **fluid** is a substance that moves under the action of a deforming force, has no rigidity (so that the shear stress vanishes everywhere).

The **fluid element** should be large enough to enclose many molecules, but small enough to cope with large gradients. By looking at the local density, we find that a fluid element should be of the order 10^{-5}m in radius.

Compressible effects in a fluid are only important if the speed of the fluid exceeds the sound speed in the fluid.

Newtonian fluids are those which have stress proportional to strain, with the constant of proportionality the **molecular viscosity**,

$$\text{stress} = \mu \times \text{strain}.$$

We use the convention for the components of the position and velocity vector,

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (u, v, w).$$

A **streamline** is the line whose tangent is everywhere parallel to $\mathbf{u}$, instantaneously. Also called lines of the flow. They are found by solving

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Streamlines cannot intersect, except at stagnation points (i.e. where the flow speed is zero).

A **streamtube** is the surface formed instantaneously by all streamlines that pass through a closed surface; allowing us to write

$$\rho u_1 S_1 = \rho u_2 S_2 \quad \Rightarrow \quad u \propto \frac{1}{S};$$

where S is the surface area of the tube with fluid having flow speed u.

The **pathlines** of a material element coincide with the streamlines in a steady flow.

A. Visualising Flows

The two main methods are:

- **Laser Doppler velocimetry**: use the Doppler beat of scattered laser light from crossed beams to deduce the velocity profile.

- **Particle image velocimetry**: seed the flow with μm particles, and illuminate with lasers. Once illuminated, snapshots are taken and analysed with computer packages.

II. THE NAVIER-STOKES EQUATIONS

We form the NS equations from a few terms:

The **material derivative** operator is defined

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla,$$

where the first term on the RHS is a global term, and the second a convective (non-linear) term.

The **viscous** term is

$$\text{viscous} : \quad \mu \nabla^2 \mathbf{u},$$

and the **pressure** term

$$\text{pressure} : \quad -\nabla p.$$

The **continuity equation** is

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.$$

For **incompressible fluids**, we have that

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = 0.$$

We combine the viscous and pressure terms to give the full **Navier-Stokes equations**,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0.$$

We also add on other body forces, such as those due to gravity.

A. The Boundary Conditions

To solve the NS, we must specify boundary conditions. The common ones are:

- *Free stream far from boundary*: $\mathbf{u} \longrightarrow U_0$.
- *Free stream on boundary*: $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$.
- *No-slip on boundary*: $\mathbf{u} \times \hat{\mathbf{n}} = 0$.

B. The Reynolds Number

We can scale the NS equations, and find the **Reynolds number**,

$$R = \frac{LU_0}{\nu},$$

where L is a length scale, and U_0 a velocity scale. Infact, one finds that

$$R = \frac{\text{inertia}}{\text{viscous}}.$$

Hence, if $R \ll 1$, viscous terms dominate the NS equations, which becomes linear. Conversely, if $R \gg 1$, inertia dominates and the NS equations becomes non-linear.

III. EXACT SOLUTIONS

We can exactly solve the NS equations for rather idealised physical systems.

A. Unidirectional Flow Between Infinite Parallel Plates

We consider fully developed flow; which means no t or x dependence on $\mathbf{u}$. Also, as the plates are infinite, no z-dependence. Hence, the NS equations become

$$\frac{\partial p}{\partial x} = -G = \mu \frac{\partial^2 u}{\partial y^2},$$

where G is the **constant pressure gradient**. This easily integrates to give the velocity profile

$$u(y) = A + By - \frac{Gy^2}{2\mu}.$$

We then use **boundary conditions** to determine the constants of integration A and B. Using no-slip, $u(y = \pm h) = 0$, gives

$$u(y) = \frac{G}{2\mu} \left(h^2 - y^2 \right).$$

This is a parabolic velocity profile.

B. Flow in a Pipe: Hagen-Poiseulle Flow

Here we consider a flow as in the previous case, but between a long circular pipe. We again use the no-slip boundary condition at $r = a$. Hence, solving the NS equations in cylindrical polars, we find the **parabolic velocity profile**

$$u_z(r) = \frac{G}{4\mu}\left(a^2 - r^2\right).$$

We can use this to compute the **volume flow rate**,

$$Q = \int_0^a u_z\, 2\pi r\, dr,$$

where we find the relations

$$Q \propto \frac{p_0 - p_1}{\ell}, \quad Q \propto a^4.$$

These dependencies on the tube length ℓ and radius a are known as **Poiseullie's law**. The volume flow rate allows computation of μ for a fluid. The law is only valid for $R < 30$, for all pipes. For any flow of Reynolds number R, for a pipe of diameter d, an **entry length** x is required to establish the steady flow used in the assumption;

$$\frac{x}{d} = \frac{R}{30}.$$

C. Flow Down an Inclined Plane

We consider a steady, fully developed flow, driven by gravity. In the x-direction, the NS equations become

$$0 = \mu\frac{\partial^2 u}{\partial y^2} + \rho g \sin\theta,$$

and in the y-direction,

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\theta.$$

The latter equation can be simply integrated to give

$$p - p_0 = \rho g\,(h - y)\cos\theta.$$

The former is easily integrated to give the semi-parabolic velocity profile,

$$u(y) = \frac{g\sin\theta}{\nu}\left(hy - \frac{1}{2}y^2\right).$$

The **average velocity** is just

$$\bar{u} = \frac{1}{h}\int_0^h u(y)\,dy = \frac{2}{3}u(y=h).$$

Hence, the relation between the surface speed $u(h)$ and average speed is just

$$u(h) = \frac{3}{2}\bar{u}.$$

The analysis only holds for flows with $R < 20$, else surface effects come into play.

D. Unsteady Flow

We consider a flow filling the upper-half of the plane, where we move the lower boundary periodically. Hence, the NS equations become

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2},$$

and we use the boundary conditions

$$u(y\to\infty) = 0, \quad u(y=0) = U_0\cos\omega t.$$

We solve the NS equation with ansatz

$$u(y,t) = f(y)e^{i\omega t},$$

and find

$$u(y,t) = U_0 e^{-ky}\cos\left(\omega t - ky\right), \quad k = \sqrt{\frac{\omega}{2\nu}}.$$

The wavenumber k is called the **propagation depth**, as it comes into the damping term preceding the oscillatory term.

E. Steady Flow With Inflow

We consider a flow bounded by two porous plates, with the lower plate moving at speed U_0. The only driving term is the movement of the fluid horizontally – the fluid moves across the porous boundaries with speed V_0. Hence, p is a constant. The NS equations become

$$\nu\frac{\partial^2 u}{\partial y^2} + V_0\frac{\partial u}{\partial y} = 0.$$

The solution gives the velocity profile,

$$u(y) = A + Be^{-V_0 y/\nu}.$$

If V_0 is small, the profile is linear – **Couette flow**. If V_0 is large, the flow is confined to the lower wall, generating a **boundary layer**.

IV. THE STREAMFUNCTION

Following the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, we can set

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{(Cartesian)}$$

Hence, we also see that

$$d\psi = 0 \quad \text{on a streamline.}$$

Indeed, ψ is a constant on a streamline. Streamlines only cross at stagnation points.

V. STOKES FLOW

For $R \ll 1$, we ignore the inertia (i.e. non-linear) terms in the NS equations, so that the NS equations become what we call the Stokes equations,

$$\nabla p = \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0.$$

As these equations are **linear**, their solutions are **reversible**. A consequence of this is that if a swimmer swims in a Stokesian flow with a symmetry, then the swimmer cannot move. However, if the swimmer breaks the symmetry, the swimmer can swim and move.

To achieve a Stokesian flow, we need a small length scale L, or large viscosity ν.

VI. STEADY INVISCID FLOW

Here we take $\nu \to 0$, which means that $R \to \infty$. In this limit, we ignore the viscous term in the NS equations (as well as the time dependent terms), to get **Euler's equation**

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p.$$

As this is now a first order equation, we tend to reject the no-slip boundary condition. Writing Euler's equation along a streamline we find,

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx},$$

which is easily integrated to give **Bernoulli's equation**,

$$\frac{1}{2}\rho U^2 + p = \text{const.}$$

This constant is the same along a given streamline.

For irrotational flow, $\nabla \times \mathbf{u} = 0$, and hence we write Bernoulli's equation as

$$\nabla\left(\frac{1}{2}\rho U^2 + p\right) = 0.$$

D'Alembert's paradox is that in such a flow, there is no pressure difference along a streamline. Hence, there is no force on an object in such a flow. Therefore, there is **no drag**. This is akin to saying that if we place a cylinder in a steady inviscid flow, the cylinder will not be carried along by the flow.

We can use Bernoulli's equation along a streamline going from the top of a tank, to a hole in the side,

$$\frac{p_{\text{top}}}{\rho} = \frac{p_{\text{hole}}}{\rho} + \frac{1}{2}u_{\text{hole}}^2.$$

Also, the pressure difference between the top and the hole,

$$p_{\text{top}} - p_{\text{hole}} = \rho g h,$$

where $h = h_{\text{top}} - h_{\text{hole}}$. Hence, putting the two together, we can deduce an expression for the speed of the fluid as it leaves the hole in the side of the tank,

$$\frac{1}{2}u_{\text{hole}}^2 = gh \quad \Rightarrow \quad u_{\text{hole}} = \sqrt{2gh}.$$

VII. VORTICITY

We define vorticity to be

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

So, in 2D flow, we only have a ω_z-component.

The **vorticity equation** is

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \nu \nabla^2 \boldsymbol{\omega}.$$

For steady flows, $\frac{\partial \boldsymbol{\omega}}{\partial t} = 0$, which means that if a flow starts having zero vorticity, it always has zero vorticity.

2D **solid body rotation** has $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$, which gives a vorticity of

$$\boldsymbol{\omega} = 2\Omega \hat{\mathbf{k}}.$$

Hele-Shaw flow is an example of thin-film irrotational flow. The NS equations are

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h-z), \quad v = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h-z),$$

so that computing $\boldsymbol{\omega}$ gives $\omega_z = 0$.

A. The Complex Potential

For **irrotational flows**, we have that $\nabla \times \mathbf{u} = 0$. Hence, we may write $\mathbf{u} = -\nabla \phi$, where ϕ is some scalar function we call the **velocity potential**. Hence,

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}.$$

Recall that for **incompressible flows**, the streamfunction ψ is such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}.$$

Hence, equating, we arrive at the **Cauchy-Riemann equations**,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Notice that both the velocity potential ϕ and streamfunction ψ are harmonic functions,

$$\nabla^2 \phi = \nabla^2 \psi = 0,$$

and that lines of equipotential and streamlines are orthogonal,

$$\nabla \phi \cdot \nabla \psi = 0.$$

Hence, we define the **complex potential** to be

$$w = \phi + i\psi.$$

Hence, $w = f(z)$, where $z = x + iy$.

Some examples:

- Flow between two boundaries with internal angle α,

$$w = Ua\left(\frac{z}{a}\right)^{\pi/\alpha},$$

- 4-roll mill,

$$w = \frac{Uz^2}{a}.$$

VIII. LUBRICATION THEORY

These concern **thin film flows**, where we choose **sensible length scales**,

$$\begin{aligned} \nu\nabla^2\mathbf{u} &\sim \mathcal{O}\left(\frac{\nu U}{h^2}\right) \quad \text{viscous,} \\ \mathbf{u}\cdot\nabla\mathbf{u} &\sim \mathcal{O}\left(\frac{U^2}{L}\right) \quad \text{inertia.} \end{aligned}$$

Hence, the ratio is just

$$\frac{\text{inertia}}{\text{viscous}} \sim \frac{UL}{\nu}\left(\frac{h}{L}\right)^2 = R\left(\frac{h}{L}\right)^2.$$

Therefore, even if R is large, if h is small enough, we can neglect the inertia term to linearise the NS equations. This is happens in the **lubrication limit**. The NS equations then become

$$-\frac{1}{\mu}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A. Cavitation

The pressure difference for an accelerating fluid is just

$$p = p_0 + \rho(g - \beta)h.$$

Hence, for a large enough acceleration β, there will be a negative pressure. Hence, vaporisation of the fluid.

IX. AEROFOILS

The upward force, per unit length, on an aerofoil is just

$$f = (p_{\mathrm{B}} - p_{\mathrm{T}})dx.$$

Considering a streamline around the aerofoil, and applying Bernoulli (considering thin aerofoils relative to their length),

$$p_{\mathrm{B}} - p_{\mathrm{T}} = \rho U_0 (u_{\mathrm{T}} - u_{\mathrm{B}}).$$

Hence, the **total lift** is

$$L = \int_0^c f\,dx = \rho U_0 \int_0^c u_{\mathrm{T}} - u_{\mathrm{B}}\,dx.$$

The **circulation** is defined to be

$$\Gamma = \oint \mathbf{u} \cdot d\boldsymbol{\ell} = \int_S \nabla \times \mathbf{u}\,d\mathbf{S},$$

where the last equality follows from Stokes' theorem. Hence, we see that **irrotational flows have zero circulation**. We can also see that

$$L = -\rho U_0 \Gamma,$$

which means that **lift implies circulation**.

Kelvin's circulation theorem is that

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt} \oint \mathbf{u} \cdot d\boldsymbol{\ell} = 0.$$

That is, **circulation is conserved**. Therefore, when lift starts, an eddy of opposite circulation begins.

The **Magnus effect** is that when a rotating cylinder is placed in a flow, there is a pressure difference, which generates a force on the cylinder.

X. BOUNDARY LAYERS

Here we choose **different length scales** of the length L and height δ for the inertia and viscous terms. Hence,

$$\begin{aligned} \text{inertia}: &\qquad \mathbf{u} \cdot \nabla \mathbf{u} \sim \frac{U^2}{L}, \\ \text{viscous}: &\qquad \nu \nabla^2 \mathbf{u} \sim \frac{\nu U}{\delta^2}. \end{aligned}$$

Hence,

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{R}}.$$

In **boundary layer flow**, the viscous terms dominate. Negative pressure gradients give an accelerated boundary layer flow; positive give decelerating flow; zero give uniform flow.

XI. TURBULENCE & INSTABILITIES

We consider the linear stability of solutions. We perturb a solution

$$A(t) = A_0 + \epsilon(t), \quad \epsilon \ll 1.$$

This perturbed solution is substituted into the equation, and solved for $\epsilon(t)$. Stability of the solution is determined via

- $\epsilon(t)$ decaying gives a **stable** solution,
- $\epsilon(t)$ gowing gives an **unstable** solution.

If there is a change in stability with some parameter λ, then we have a **bifurcation**. Effectively, one stable solution becomes two; the choice of solution is essentially random. We conventionally call the solution $A = 0$ the **trivial state**.

We can get **super-critical** and **subcritical** pitchfork bifurcations, and **Hopf bifurcations**: **periodic oscillation between stable states**.

A. Onset of Turbulence

Turbulence is the random unpredictable motion of a fluid.

The **Landau theory** for the onset of turbulence is that an infinite sequence of instabilities excite an infinite number of modes, at finite R.

The **chaotic theory** is that a finite number of instabilities produce turbulence.

B. Rayleigh's Criterion

This considers a rotating flow, where we take toroidal elements of the flow.

If we displace the toroid, the stability of the flow can be determined by considering the circulation $\Gamma_i = r_i v_i$:

- If $\Gamma_2^2 > \Gamma_1^2$, the flow is **stable**,
- If $\Gamma_2^2 < \Gamma_1^2$, the flow is **unstable**.

C. Energy Cascade & The Kolmogorov Spectrum

We consider energy on large scales going to smaller scales, until viscosity takes over on the smallest scales. Hence, as the energy cascades down there is a greater complexity of the flow. Vortex lines stretch, increasing the vorticity.

We define the average kinetic energy per unit volume

$$\phi = \frac{1}{2}\rho\bar{q}^2,$$

so that

$$\frac{\phi}{\rho} = \int_0^\infty E(k)dk.$$

The **energy dissipation** $\mathcal{E}$ is the energy flow per unit mass per unit time, and has units $[\mathcal{E}] = L^2T^{-3}$. Hence, given $[E] = L^3T^{-2}$, and the ansatz

$$E \propto \mathcal{E}^\alpha k^\beta,$$

we find by comparing dimensions

$$E(\mathcal{E}, k) = C\mathcal{E}^{2/3}k^{-5/3}.$$

This compares well with experiment, up to high k (i.e. very small scales).

D. Reynold's Stresses

Basically, if we introduce a term to the velocity which has zero mean, the average momentum has an extra term due to fluctuations, and one finds

$$\text{Reynold's stresses} \approx 10^{-3} \times R \times \text{viscous forces}.$$