Physics of Fluids

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Abstract

These are a set of notes I have made, based on lectures given by T.Mullin at the University of Manchester Jan-May '09. Please e-mail me with any comments/corrections: jon@jpoffline.com. These notes may be found at www.jpoffline.com.

Contents

1 Introduction

The aim of this is to discuss classical fluid dynamics, with some emphasis upon examples & applications. We shall vaguely derive most equations – but these derivations will not be rigorous!

We seek some universality – typical behaviour from non-linear systems. In particular, how the onset of turbulence occurs in a system. Consider a system, whereby a smooth input is taken by a system, which has non-linear deterministic equations of motion, and chaos is outputted. This chaotic output is still deterministic, and if the initial conditions are known precisely (i.e. with no error at all), then this chaotic output will be entirely predictable. However, that initial conditions are only known to a given accuracy, the output will always (practically) be unpredictable: chaos. A very simple example of such a system is the undamped pendulum. Its full equation of motion is of the form

$$\ddot{\theta} + \sin\theta = 0.$$

Now, this is a highly non-linear problem. The usual route for solving is to make the small angle approximation ($\sin\theta \to \theta$), to give the easily soluble linear equation

$$\ddot{\theta} + \theta = 0.$$

So, for large initial amplitude (i.e. where the small angle approximation breaks down), the equations of motion are non-linear.

Another example is the Lorenz system, whereby the governing differential equations are

$$\begin{aligned} \dot{x} &= \sigma(x - y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy; \end{aligned}$$

where σ, r are parameters. Such differential equations produce the so-called "Lorenz attractor", and the ideas behind them are able to model a variety of systems; such as weather modeling.

Now, it is usually easy to write down the equations of motion of a non-linear system, but it is virtually impossible to solve them, even with modern supercomputers.

1.1 Fluid as a Continuum

We shall define that a fluid is some substance that moves under the action of a deforming force, with true fluids having no rigidity (so that they are unable to support shear stress anywhere – shear stress vanishes everywhere).

Not all materials which fall into the first category are fluids. For example, thixotropic and viscoelastic solids are not fluids (with examples being paint, jellies, toothpaste and putty, respectively). However, others are – examples being liquid glass and pitch/tar.

Flowing sand is "fluid-like", but it can support shear stress in some circumstances. Consider a grain silo, with a small open tube at the bottom, as in Figure (1.1). The pressure in the silo is *independent of depth* (note that this is in contrast to water, whose pressure is dependent upon depth below the surface); and the rate of flow of the grain, through the tube is entirely determined by frictional effects at the tube (again, the rate of flow of water through such a tube is dependent upon the amount of water above the tube). We will consider both liquids

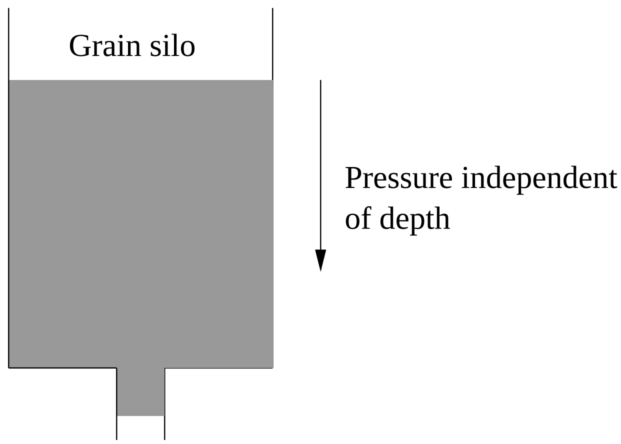


Figure 1.1: A grain silo: the pressure within the silo is independent of depth below the surface; this is in complete contrast with a tub filled with water.

& gasses, as the dynamics of both is determined by the same set of equations.

1.1.1 Microscopic Level & The Continuum

Liquid molecules are in intimate contact with one another. Thus, if we shear such an object, molecules will realign in a very complicated manner. Gases, on the other hand, are composed of well separated molecules, whose mean free path is large.

Basically, fluids are very difficult (if not impossible) to treat at a microscopic level. So, to get around this, we introduce the *hypothesis* of a *continuum*.

We introduce "fluid elements" as being large enough to enclose many molecules, but small enough to cope with large gradients. Hence, fluid dynamics is concerned with macroscopic flow phenomena.

The first of the requirements implies that average quantities are well defined – such as temperature, pressure, velocity, density.

If we were to plot the dependance of the average density $\bar{\rho}$ within a cube of side a, we find that at very small scales ($\approx 10^{-10}$m), the average density is ill defined, with many "random fluctuations (this is due to molecular fluctuations). On scales $\approx 10^{-7} \to 10^{-4}$m, we see that $\bar{\rho}$ is approximately constant, and above 10^{-1}m, either raise or drop. Now, the reasons behind raising or lowering is that the density changes due to large scale effects, such as temperature changes. The intermediate, roughly constant value of $\bar{\rho}$ is the *local value*, and is taken to be the density.

Thus, the size of a (i.e. the size of the "fluid element") should be $\approx 10^{-5}$m, in order that there is little variation in physical and dynamical properties within the element. Hence, an instrument sensitive to $(10^{-5}\text{m})^3 = 10^{-15}\text{m}^{-3}$ will measure a local value. For example, in such a volume will be $\approx 10^{10}$ molecules of air.

Molecules can be exchanged between "fluid particles" by the diffusion process – via viscosity (in which case momentum will be transferred) or thermal conductivity (temperature). So, the size of the fluid particles must be greater than the mean free path of the molecules; that is, molecules must suffer many collisions. The mean free path, of air molecules at STP is $\approx 10^{-7}$m.

From hereon, the concept of a "point" in a fluid, and "fluid particle" are taken to be the same.

We shall only consider *incompressible Newtonian flows*. Now, compressible effects are only important when the speed of the fluid exceeds the speed of sound in the fluid;

$$c_\text{s}^\text{air} \approx 340\text{ms}^{-1}, \quad c_\text{s}^\text{water} \approx 1400\text{ms}^{-1}.$$

Hence, only for things like underwater explosions, or air passing over a jets wings, will we have to consider compressible effects (which we shall not do here). Newtonian fluids are fluids for whom stress is proportional to strain, with the constant of proportionality being the molecular viscosity;

$$\text{Newtonain fluid} \quad \Rightarrow \quad \text{stress} = \mu \times \text{strain}.$$

For example, if a fluid element is subject to a strain of $\frac{\partial u}{\partial y}$ (i.e. moves along one edge), then

$$\tau = \mu \frac{\partial u}{\partial y}.$$

Most fluids are Newtonian, but a few examples that are non-Newtonian include polymer solutions (such as shower gel, shampoo etc) and liquid crystals (as found in laptop display screens).

2 Developing the Equations of Motion

Here we shall present a rather heuristic derivation of the Navier-Stokes equations, and introduce some other useful concepts.

We use the notation that the position vector $\mathbf{x} = (x, y, z)$ and velocity vector $\mathbf{u} = (u, v, w)$, where $u = u(x, y, z)$ etc.

2.1 Streamlines

A streamline in a fluid is the line whose tangent is everywhere parallel to $\mathbf{u}$ instantaneously, and is also called a line of flow. The family of streamlines are solutions, at some instant, of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{2.1}$$

Streamlines cannot intersect, except at positions of zero velocity (otherwise there will be a multiply defined direction of velocity). In steady flow, streamlines have the same form for all times.

A streamtube is the surface formed instantaneously by all streamlines that pass through a give closed curve. If q_i is the speed of the fluid, which has density ρ that goes through a surface of area S_i, then conservation of mass flux determines that

$$\rho q_1 S_1 = \rho q_2 S_2. \tag{2.2}$$

Hence, this easily gives us

$$\frac{q_2}{q_1} = \frac{S_1}{S_2} \quad \Rightarrow \quad q \propto \frac{1}{S}.$$

That is, speed is inversely proportional to surface area. Hence, closely spaced streamlines mark positions of fast moving flow (and vice versa). Also, divergence of streamlines mean that the flow is accelerating; convergence that the flow is decelerating.

A pathline of a material element coincides with a streamline in a steady flow. A streakline formed from releasing dye (for example) in a fluid, coincides with a pathline in a steady flow. Such a visualisation tool is useful in 2D flow, but is near impossible in 3D flow.

2.1.1 Visualising & Measuring Flows

We can visualise a flow using smoke, dyes or anisotropic particles, to name but a few. Doing so gives a general insight into the structure of the flow fields, which gives qualitative information on the flow.

We can make qualitative measurements on the flow (such as the flows pressure, speeds, velocity, velocity distribution) using Laser Doppler Velocimetry (LDV) or Particle Image Velocimetry (PIV).

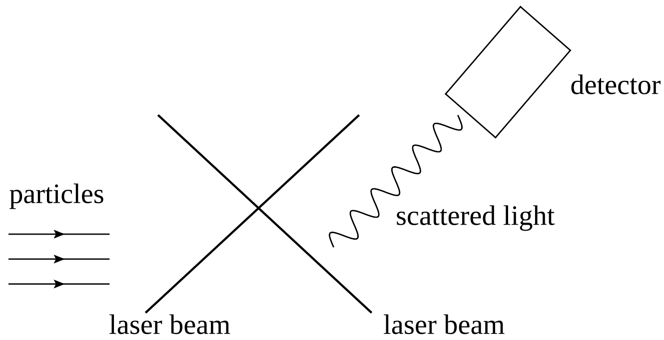


Figure 2.1: A schematic setup of Laser Doppler Velocimetry. Particles of the flow are incident upon crossed laser beams. Scattered light is then picked up by a detector, and analysed by assessing its Doppler shift.

With reference to Figure (2.1), we see a rough setup of LDV. Practically, one uses the Doppler beat (upon mixing with a reference source), so that

$$f_{\text{dopp}} \propto U,$$

the speed of the flow.

PIV seeds a flow with small (of the order μm) particles, and illuminates the flow with high power lasers. Successive snapshots are then taken, and software is used to track particles & plot velocity distributions etc.

2.2 The Navier-Stokes Equations

These equations arise from Newtons second law.

For a unit volume of fluid, the rough form of the Navier-Stokes equations is

$$\rho \frac{D\mathbf{u}}{Dt} = F + P, \tag{2.3}$$

where F is the body force on the unit of fluid (due to external influences, for example gravity or electrodynamic forces), and P the pressure on the fluid (due to internal influences, for example viscous forces and pressure). Body forces tend to be long range, and constant over the fluid, but pressure forces depend on the rate at which the fluid is strained, and tends to be short range and molecular in origin. In general, for internal flows, there is a hydrostatic balance between body forces and pressure forces, for a fluid at rest. Hence, we shall only consider dynamic pressure forces due to fluid motion. Other forces could be those due to thermal, coriolis or chemical. This is not true if gravity acts as a restoring force. Other effects, such as surface tension may also be important.

The first term in (2.3) is the so called substantive derivative, or the material derivative.

2.2.1 The Material Derivative

We define an operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{2.4}$$

to be the material derivative. So, for example,

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \tag{2.5}$$

which has the three components

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} = \quad & \hat{\mathbf{x}} \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) \\ & + \hat{\mathbf{y}} \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) \\ & + \hat{\mathbf{z}} \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z} \right). \end{aligned}$$

In index notation, (2.5) reads

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}.$$

To understand the material derivative a bit better, or to get an intuition as to its meaning, one could consider a flow with

$$\mathbf{u} = (u, v, w) = \Omega(-y, x, 0),$$

which corresponds to a uniform rotation. Then,

$$\mathbf{u} \cdot \nabla \mathbf{u} = \hat{\mathbf{x}}(-\Omega^2 x) + \hat{\mathbf{y}}(-\Omega^2 y) = -\Omega^2(x, y),$$

which is a centrifugal acceleration $\Omega^2 \mathbf{r}$ towards the centre of a circle.

The material derivative reflects the fact that fluid particles move from one location to another in the flow, and can be accelerated/decellerated by both movement to a place of higher velocity as well as a temporal change in the global flow field. That is, fluid can be accelerated in a steady flow.

One can think of the $\frac{\partial}{\partial t}$ term as a global derivative, and $\mathbf{u} \cdot \nabla$ as a convective derivative.

For example, one could consider an aerofoil in uniform flow. The fluid flows faster over the top than the bottom, which causes a change in pressure between the top and bottom, which causes lift.

2.2.2 Developing the Navier-Stokes Equations

We shall consider incompressible, viscous fluids, with pressure forces and the continuity equation.

Viscous Forces Consider a "box of fluid" $ABCD$, which is in a flow. Suppose that the height of the box is δy. Then, the viscous forces acting in the top section of the box, in the x-direction, per unit area, is

$$\mu \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \delta x \delta z.$$

Hence, the net viscous forces on a box is

$$\left[\mu \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}\right] \delta x \delta z.$$

If we let $\delta y \to 0$, then this goes to a partial derivative, and so

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \delta x \delta y \delta z.$$

Hence, the net viscous forces, per unit volume, on a fluid element in the x-direction, is

$$\mu \frac{\partial^2 u}{\partial y^2}.$$

In a similar way, one can find the total viscous forces, in the x-direction, to be

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \mu \nabla^2 u. \tag{2.6}$$

Hence, the viscous forces are given by

$$\mu \nabla^2 \mathbf{u}, \tag{2.7}$$

which is linear in $\mathbf{u}$. This is a diffusion term, where momentum is diffused into heat by the action of viscosity – a kinematic force.

Pressure Forces The net pressure acting on an element in the downstream direction, is given by

$$(p_x - p_{x+\delta x})\, \delta x \delta z.$$

But,

$$p_{x+\delta x} \delta y \delta z = \left(p_x + \frac{\partial p}{\partial x}\right) \delta y \delta z.$$

Thus, the net pressure is

$$-\frac{\partial p}{\partial x}\delta x\delta y\delta z,$$

or

$$-\frac{\partial p}{\partial x}$$

per unit volume. Thus, in total, the net pressure forces, per unit volume, are given by

$$-\nabla p. \tag{2.8}$$

Thus, notice that pressure acts in opposition to viscous forces.

Hence, the equations of motion are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{2.9}$$

where the molecular viscosity μ is related to the density ρ via the kinematic viscosity ν as

$$\nu \equiv \frac{\mu}{\rho}. \tag{2.10}$$

Notice that using the material derivative and index notation, (2.9) reads

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}.$$

We tabulate values of μ and ν for water and air, to get a feel for the numbers.

	$\mu(\text{kg m}^{-1}\text{s}^{-1})$	$\nu(\text{mm}^2\text{s}^{-1})$
water	10×10^{-2}	1
air	1.3×10^{-3}	15

By way of units, one centi-Stoke is defined as

$$1\text{cS} = 1\text{mm}^2\text{s}^{-1}.$$

Notice that water has about 15 times the molecular viscosity of air; but air has about 15 times the kinematic viscosity of water (i.e. the molecular viscosity tells you how "gooey" the fluid is, but the kinematic viscosity tells you about how it flows). Infact, in fluid dynamics, the kinematic viscosity ν is more useful than the molecular viscosity μ when determining the dynamical state of the flow.

The Continuity Equation We have three equations (one for each component of (2.9)) for four unknowns (the three velocity components, and density). Hence, we need a fourth equation. We appeal to a continuity equation.

Consider the rate of change of total mass outflux through some elemental surface,

$$\int \frac{\partial \rho}{\partial t} dV = -\int \rho \mathbf{u} \cdot \mathbf{n} dA.$$

Then, using the divergence theorem, this easily becomes

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV = 0.$$

Hence, as this is true for any volume, and indeed zero volume, the integrand must be zero. Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.11}$$

If we expand out the divergence, we get

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0,$$

where we note the material derivative of the density ρ is present in the first and third terms. Hence,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{2.12}$$

This is the continuity equation. For an incompressible fluid,

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = 0.$$

Navier-Stokes Equations Hence, we can write the Navier-Stokes equations, after our rather wooly derivation. They are composed of the material derivative and continuity equation. Thus:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}, \tag{2.13}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2.14}$$

These are non-linear differential equations. The non-linearity comes from the second term from the left, on the first equation. There are three equations in the first, and a fourth in the second; hence, in full, the Navier-Stokes equations (in Cartesian coordinates) are

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

We will mainly be concerned in deducing which terms go to zero, so that we can produce a "simple" analytic solution for a given situation. Non-linearity also enters in the boundary conditions. The common boundary conditions are:

- Free stream far from boundaries:
$$\mathbf{u} = U_0.$$
- Free stream at solid boundary:
$$\mathbf{u}\cdot\mathbf{n} = 0.$$
- No-slip:
$$\mathbf{u}\times\mathbf{n} = 0.$$

The last boundary condition, the no-slip condition, cannot be justified on a molecular level, but is very well supported by numerical and physical experiments. However, problems do arise in gas dynamics, where one has so-called "Knudsen layers of slip".

2.3 Scaling Navier-Stokes & Reynolds Number

Consider a steady flow. Then, (2.13) reads

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u}. \tag{2.15}$$

Now, we can introduce "new scales" for length and speed, so that

$$x^* = x/L, \quad y^* = y/L, \quad z^* = z/L, \quad \mathbf{u}^* = \mathbf{u}/U_0, \quad \Delta p^* = \Delta p/\rho U_0^2.$$

The last quantity, Δp is the difference between actual pressure, and some reference. Hence, all "starred" quantities are dimensionless. Hence, using these in (2.15) gives

$$\frac{U_0^2}{L}\left(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^*\right) = -\frac{U_0^2}{L}\nabla^*\left(\Delta p^*\right)+\frac{\nu U_0}{L^2}(\nabla^*)^2\mathbf{u}^*,$$

or,

$$(\mathbf{u}^* \cdot \nabla^* \mathbf{u}^*) = -\nabla^* (\Delta p^*) + \frac{1}{R} \nabla^{*2} \mathbf{u}^*,$$

where

$$R \equiv \frac{LU_0}{\nu} \qquad (2.16)$$

is the Reynolds number. This has the interpretation of telling you how big the inertia forces are, relative to the viscous forces;

$$R = \frac{\text{inertia}}{\text{viscous}}.$$

Thus, if $R > 1$, the inertia forces dominate over viscous forces; and if $R < 1$, then viscous forces dominates inertia. Notice that it is the inertia term which has the non-linearity in the NS equations. Hence, one can see that if $R \ll 1$, the NS equations become more linear,

$$\longrightarrow \nabla p = \mu \nabla^2 \mathbf{u}.$$

If $R \to \infty$, then the equations become non-linear due to the inertia term $\mathbf{u} \cdot \nabla \mathbf{u}$:

$$\longrightarrow \frac{D\mathbf{u}}{Dt} = \frac{1}{\rho} \nabla p.$$

Some examples:

System	**Reynolds Number (approx)**
ball bearing in syrup	1
continental drift	10^{-20}
biological organism	10^{-2}
QE2 cruising	10^{10}
Beoing 747 cruising	10^{8}
whale	10^{4}
hailstone	10^{4}
finger in bath	10^{3}

Hence, one can see that everything except the first three systems are non-linear. The first three systems have solutions which are reversible (as linear), and viscous effects carry over long distances. The non-linear equations dont tend to be reversible.

3 Exact Solutions

Here we shall consider a few cases, and solve the Navier-Stokes equations exactly.

3.1 Unidirectional Flow Between Infinite Plates

Let us first consider steady unidirectional flow between a pair of infinite parallel plates, a distance $2h$ apart, with motion being a result of a constant pressure gradient. We shall arrange our coordinate system such that y points "up" the plates, x points "along" the plates and z points "between" the plates (that the plates are infinite, means that z stretches off to infinity). So, the Navier-Stokes equation (2.13)

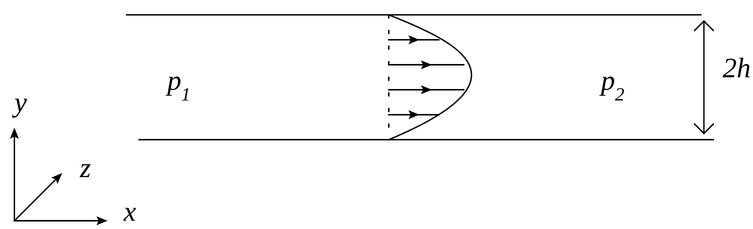


Figure 3.1: The general setup for flow between plates. Also shown is the derived velocity profile for $p_1 > p_2$.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u},$$

or, in the x-direction (also expanding out the $\mathbf{u} \cdot \nabla$ operator),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

Now, as the flow is steady (also called "fully-developed"), there is no t-dependance, and so the first term on the LHS is zero. As the plates are infinite, there is no z dependance – which kills off the fourth and eighth terms. Also, there is no x dependance of the flow speed u, again, as infinite – this kills off the second and sixth terms. Hence, the above NS equation simplifies to

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} = -G, \tag{3.1}$$

where we have used the fact that there is a constant pressure gradient, which we set to $-G$. We have also set $v = 0$; to see why, consider the 2D incompressibility statement,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Now, suppose we introduce scales,

$$u \sim U, v \sim V, x \sim L, y \sim h.$$

Then, the "scaled" incompressibility equation reads

$$\frac{U}{L} + \frac{V}{h} = 0 \quad \Rightarrow \quad V \sim \frac{Uh}{L}.$$

The last expression follows as we require both terms in the first expression to balance in orders. Now, as $L \gg h$ in our setup (infact, basically in everything we do), we have that $V \sim 0$. Therefore, we say that by dimensional scaling arguments, we set $v = 0$.

We can integrate (3.1), to see that

$$u(y) = A + By - \frac{Gy^2}{2\mu}, \tag{3.2}$$

where A and B are constants of integration. Now, to go further, we need boundary conditions, so that we can determine A and B. Let us impose that the flow is stationary at the boundaries (i.e. the no-slip boundary condition). This corresponds to

$$u = 0 \quad \text{at} \quad y = \pm h.$$

So,

$$u(h) = A + Bh - \frac{Gh^2}{2\mu} = 0 \qquad \Rightarrow \qquad A = -Bh + \frac{Gh^2}{2\mu},$$
$$u(-h) = A - Bh - \frac{Gh^2}{2\mu} = 0 \qquad \Rightarrow \qquad A = Bh + \frac{Gh^2}{2\mu},$$

which is only satisfied by

$$B = 0,$$

in which case

$$A = \frac{Gh^2}{2\mu}.$$

Hence, using these constants in the solution (3.2), we see that

$$u(y) = \frac{G}{2\mu}\left(h^2 - y^2\right).$$

Thus, the velocity profile is parabolic, as in Figure (3.1). One can easily show that

$$u_{\max} = \frac{Gh^2}{2\mu},$$

which is the speed in the middle of the plates. Now, this solution becomes unstable at $R = 5772$; that is, below this Reynolds number, experiment confirms this prediction very well, but, above, non-linearity sets in and turbulence is observed.

3.2 Hagen-Poiseuille Flow

Now consider fully developed steady flow in a long circular pipe, with a constant pressure gradient along the pipe.

We setup the cylindrical polar coordinate system; so that r is the distance from the centre, z the distance down, and θ an angle around the pipe.

We assume that the flow is of the form

$$\mathbf{u} = (0, 0, u_z(r)),$$

only. Also, by the definition of the problem,

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0.$$

Recall the expressions for the grad, divergence and Laplacian, in cylindrical polar coordinates,

$$\begin{aligned}
\nabla &= \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right), \\
\nabla \cdot \mathbf{a} &= \frac{1}{r}\frac{\partial}{\partial r}(r a_r) + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z} \\
&= \frac{a_r}{r} + \frac{\partial a_r}{\partial r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}, \\
\nabla^2 \mathbf{a} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \mathbf{a}}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \mathbf{a}}{\partial \theta^2} + \frac{\partial^2 \mathbf{a}}{\partial z^2} \\
&= \frac{\partial^2 \mathbf{a}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{a}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathbf{a}}{\partial \theta^2} + \frac{\partial^2 \mathbf{a}}{\partial z^2}.
\end{aligned}$$

Now, as we only have a u_z component, the Navier-Stokes equation (for z) becomes

$$\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right).$$

We can now simplify this, by noting that there is no time dependance, and $\mathbf{u} = (0, 0, u_z(r))$ only, so that

$$0 = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} \right),$$

also noting the constant pressure gradient

$$\frac{\partial p}{\partial z} = -G,$$

hence, the NS equation becomes

$$-\frac{Gr}{\mu} = \frac{d}{dr}\left(r \frac{du_z}{dr} \right).$$

We can now integrate this once,

$$-\frac{Gr^2}{2\mu} = r\frac{du_z}{dr} + A,$$

but, if we divide through by r, then we see that

$$\frac{A}{r} - \frac{Gr}{2\mu} = \frac{du_z}{dr}.$$

Now, at $r = 0$, we have a singularity. Therefore, to get around this problem, we set $A = 0$. Hence,

$$-\frac{Gr}{2\mu} = \frac{du_z}{dr},$$

which is integrated to give

$$u_z(r) = -\frac{Gr^2}{4\mu} + B.$$

We then impose the boundary condition that $u_z(r = a) = 0$ (where a is the radius of the pipe), so that

$$B = \frac{Ga^2}{4\mu},$$

and hence the solution

$$u_z = \frac{G}{4\mu}(a^2 - r^2).$$

This is a parabolic velocity profile again.

Now, **volume flow rate** is computed via

$$Q = \int_0^a u_z 2\pi r\, dr = \frac{\pi G a^4}{8\mu}. \tag{3.3}$$

If we take the pressure gradient to be

$$G = \frac{p_0 - p_1}{l},$$

where l is the length of some pipe, and p_i are the pressures at either end, then we see that volume flow rate is

$$Q \propto \frac{p_0 - p_1}{l}, \quad Q \propto a^4. \tag{3.4}$$

These dependancies define *Poiseuille's law.* Such an expression as (3.3) allows computation of the viscosity μ of a fluid, as flow rate Q and radius a are relatively simple to measure.

Now, Poiseuille's law is only accurate for $R < 30$ (for all pipes). For greater Reynolds number R, the pipe must be "long enough" to establish an x-independent solution, otherwise the entrance of the fluid into the pipe effects the results. The general rule is

$$\frac{x}{d} = \frac{R}{30}, \tag{3.5}$$

so that for a flow with Reynolds number R, through a pipe of diameter d, then the length of the pipe needs to be x in order that Poiseuille's law holds. For example, if $R = 1800$, one needs 60 diameters length of a pipe such that Poiseuille's law can be applied.

Notice that as the Poiseuille effect has been experimentally verified, then the no-slip boundary condition is valid, as it was used in the derivation.

3.3 Flow Down Inclined Plane

Consider a steady flow down an inclined plane, driven by gravity. One can assume that the planes are infinitely long (so independent of z and x). An example of this, is water flowing down a hill, where the "upper surface" is air.

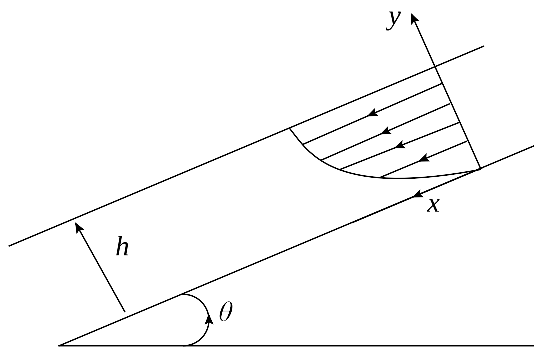


Figure 3.2: The general setup for flow down an inclined plane. Also shown is the derived semi-parabolic velocity distribution.

In the x-direction, the LHS of the NS equations is just zero, the pressure force (due to gravity) only acts in the x-direction, and the velocity only depends upon y. Hence, the NS equation in the x-direction is

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta,$$

where the last term is a "body force"; but, $\frac{\partial p}{\partial x} = 0$ (by an argument that is about to follow), so that we have

$$0 = \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta, \tag{3.6}$$

In the y-direction, we have

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos \theta,$$

which allows us to find that

$$p(y) = -\rho g y \cos \theta + A,$$

but, at $y = h$, $p = p_0$, the atmospheric pressure. Hence

$$p - p_0 = \rho g(h - y)\cos\theta,$$

which is the reason we said that $\frac{\partial p}{\partial x} = 0$. The boundary conditions we use are no slip on the surface. Thus, $u = 0$ at $y = 0$. Also, there is to be no shear stress, in which case

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0,$$

but the first term is zero, so

$$\frac{\partial u}{\partial y} = 0$$

at $y = h$. From (3.6),

$$\mu\frac{\partial^2 u}{\partial y^2} = -\rho g\sin\theta,$$

and so integrating gives

$$\mu\frac{\partial u}{\partial y} = -\rho g y\sin\theta + A,$$

but, by the free surface boundary condition, the LHS is zero at $y = h$, so that

$$A = \rho g h\sin\theta.$$

Hence,

$$\frac{\partial u}{\partial y} = \frac{g\sin\theta}{\nu}(h - y),$$

integrating again,

$$u(y) = \frac{g\sin\theta}{\nu}\left(hy - \frac{1}{2}y^2\right) + B,$$

but, using $u(y = 0) = 0$, one sees that $B = 0$. Hence, the velocity profile is

$$u(y) = \frac{g\sin\theta}{\nu}\left(hy - \frac{1}{2}y^2\right).$$

This profile is a semi-parabola, as in Figure (3.2).

Now, the average velocity flow over the layer is

$$\bar{u} = \frac{1}{h}\int_0^h u\,dy = \frac{1}{3}\frac{g\sin\theta}{\nu}h^2.$$

Notice that the surface speed, $u(y = h)$ is

$$u_{\text{surface}} = u(h) = \frac{1}{2}\frac{g\sin\theta}{\nu}h^2,$$

and hence that

$$u(h) = \frac{3}{2}\bar{u}.$$

Again, all of the above is for $R < 20$. Otherwise, non-linear effects come into play – such as "roll waves" on the surface. Considering rain flowing down a road. A typical speed is $u = 10\text{cms}^{-1}$, with depth $d = 2\text{mm}$ and kinematic viscosity $\nu = 1\text{cS}$. This gives $R = 200$, and so one would expect (and indeed sees) to see roll waves.

3.4 Unsteady Flow

Let us consider a time dependent flow. Let a viscous fluid fill the half plane above a boundary, and let the boundary move in a tangential direction, periodically; as in Figure (3.3).

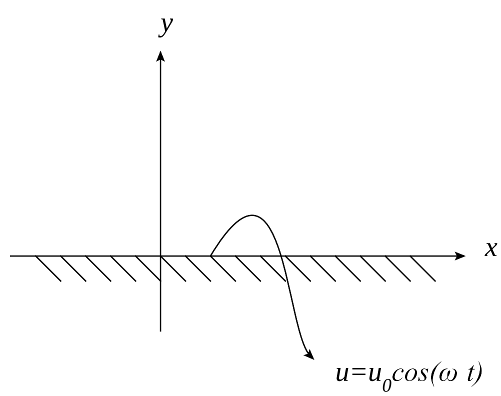


Figure 3.3: The general setup for an unsteady flow. A fluid fills the plane above some boundary, where the boundary moves periodically such that $u = u_0 \cos \omega t$.

We shall assume that everything is uniform, in the x-direction, and no z-dependance (so $v = w = 0$). Hence, the NS equation, in the x-direction, is just

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{3.7}$$

We use the boundary condition that as $y \to \infty$, $u \to 0$. Also, we have that at $y = 0$, $u(y = 0) = u_0 \cos \omega t = \Re(u_0 e^{i\omega t})$. Hence, we shall seek a solution of the form

$$u(y, t) = f(y)e^{i\omega t}.$$

So, using this, (3.7) becomes

$$\frac{i\omega}{\nu} f = \frac{d^2 f}{dy^2}. \tag{3.8}$$

If we denote

$$\alpha^2 \equiv \frac{i\omega}{\nu},$$

then the solution to (3.8) is

$$f(y) = Ae^{\alpha y} + Be^{-\alpha y}.$$

Now, our requirement is that the velocity should not diverge as $y \to \infty$; hence, we require that $A = 0$. Also, at $y = 0$, we require $f(y = 0) = u_0$ by the boundary condition. Hence,

$$f(y) = u_0 e^{\left(\frac{i\omega}{\nu}\right)^{1/2} y},$$

and therefore,

$$u(y,t) = u_0 e^{\left(\frac{i\omega}{\nu}\right)^{1/2} y} e^{i\omega t}. \tag{3.9}$$

Now, if we use the identity that

$$\sqrt{i} = \frac{1+i}{\sqrt{2}},$$

and the definition that

$$k \equiv \sqrt{\frac{\omega}{2\nu}},$$

then we see that (3.9) can be written as

$$u(y,t) = u_0 e^{-(1+i)ky} e^{i\omega t} = u_0 e^{-ky} e^{i(\omega t - ky)},$$

which is a damped travelling wave, propagating in the y-direction. Recall that we must take the real part of this expression, so that

$$u(y,t) = u_0 e^{-ky} \cos(\omega t - ky). \tag{3.10}$$

Notice that at $y = 1/k$, the amplitude of the oscillation has dropped to 1 e^{th} of its original value. Thus, we call k the *propagation depth*.

Therefore, we see that the velocity profile is a damped transverse wave of wavelength

$$\lambda = 2\pi \left(\frac{\omega}{2\nu}\right)^{-1/2},$$

propagating in the y-direction, with phase velocity

$$v_{\mathrm{p}} = \frac{\omega}{k} = (2\nu\omega)^{1/2}.$$

An example of such a system is that of the sun heating the earth, based upon a 24hr period of oscillation. This gives a penetration depth of $k = 1$m, if we use an average thermal diffusivity of soil $\approx 10^{-6}\mathrm{m}^{-2}\mathrm{s}^{-1}$.

This analysis only holds for unbounded domains. In bounded domains, large-scale circulation occurs – known as secondary streaming.

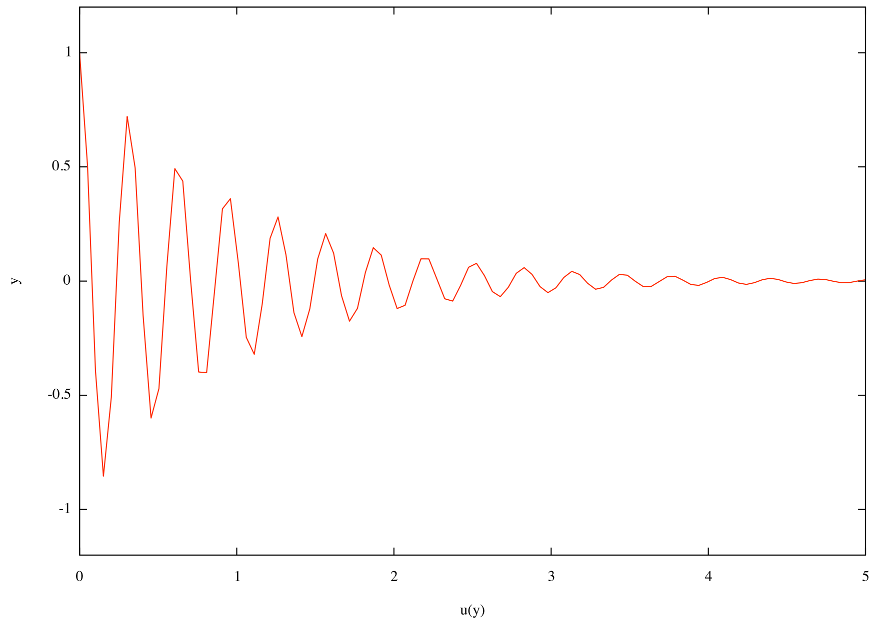


Figure 3.4: The derived solution (3.10) for unsteady flow. The boundary is on the LHS, with wave propagating from left to right.

3.5 Steady Driven Flow With Inflow

Consider viscous flow between rigid planes at $y = 0, h$, where the lower plane moves at constant speed u_0 in the x-direction, and the walls (i.e. the planes) are porous, allowing (and indeed, we enforce) a constant vertical flow v_0. See Figure (3.5).

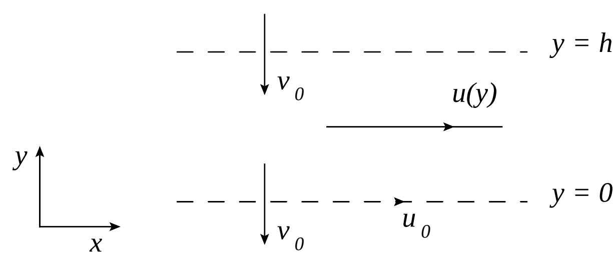


Figure 3.5: The general setup for flow between two porous planes.

We make the plates infinite in extent into the z-direction, so that

$$\mathbf{u} = (u(y), v(y), 0)$$

only. Notice that we do not have any x-dependance upon the velocity; this is because we are considering fully developed flow.

Notice that we can write the incompressibility law,

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} = 0,$$

where the second equality follows as u is not a function of x, and only of y. Hence, integrating, we see that $v = \text{const}$, and we set $v = -v_0$ (with a minus sign as the direction of flow is "down", opposite to the direction of $+y$).

The steady Navier-Stokes equations are

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u},$$

so that writing out its two components, we have

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}, \tag{3.11}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial^2 v}{\partial y^2}. \tag{3.12}$$

The first and fourth terms of the first expression are zero (as no x-dependance of u). Also, the first, second, fourth and fifth terms of the second expression are zero (as no x-dependance of v and v is a constant). So, (3.11) becomes

$$-v_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}. \tag{3.13}$$

And we see that (3.12) becomes

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad \Rightarrow \quad p = p(x).$$

If we impose that $p = p_0$ at $x = \pm\infty$, then we also have that $p = \text{const}$, so that motion is driven purely by the fluid flow, and not by a pressure gradient. Hence, (3.13) becomes

$$\nu \frac{\partial^2 u}{\partial y^2} + v_0 \frac{\partial u}{\partial y} = 0.$$

This has the easily verifiable solution,

$$u(y) = A + B e^{-\frac{v_0 y}{\nu}}.$$

If we now impose boundary conditions, we can find the values of the constants A, B. We have the no-slip boundary condition that $u = 0$ on $y = h$, which gives us

$$A = -B e^{-\frac{v_0 h}{\nu}} \quad \Rightarrow \quad u(y) = B \left(e^{-\frac{v_0 y}{\nu}} - e^{-\frac{v_0 h}{\nu}} \right).$$

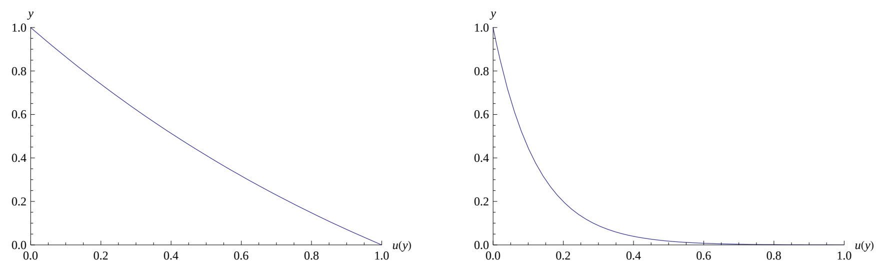
We also have that $u = u_0$ on $y = 0$. Hence,

$$u_0 = B\left(1 - e^{-\frac{v_0 h}{\nu}}\right) \quad \Rightarrow \quad B = \frac{u_0}{1 - e^{-\frac{v_0 h}{\nu}}},$$

and therefore, the velocity profile is given by

$$u(y) = u_0 \frac{e^{-\frac{v_0 y}{\nu}} - e^{-\frac{v_0 h}{\nu}}}{1 - e^{-\frac{v_0 h}{\nu}}}. \tag{3.14}$$

See Figure (3.6) for plots of the solution, in the cases of v_0 small and large – (a) and (b), respectively.



(a) v_0 small. (b) v_0 large.

Figure 3.6: Two limiting cases of the solution (3.14) to the steady driven flow with porous boundaries; in these plots, the moving boundary is along the x-axis. We see in (a) that the velocity profile approaches linear, when the speed of the inflow is very small. In (b) we see that the profile shifts most of the motion towards the moving wall, an example of a boundary layer.

Consider taking $v_0 h/\nu \ll 1$ in (3.14), expanding the exponentials,

$$u(y) \approx u_0 \frac{1 - \frac{v_0 y}{\nu} - \left(1 - \frac{v_0 h}{\nu}\right)}{1 - \left(1 - \frac{v_0 h}{\nu}\right)} = u_0\left(1 - \frac{y}{h}\right),$$

which is a linear profile, as in Figure (3.6)a. This is a linear shear flow, also called Courette flow. The other limit we can take, in (3.14) is v_0 large, in which case

$$u(y) \approx u_0 e^{-\frac{v_0 y}{\nu}},$$

which effectively confines the flow to the lower (moving) wall. That is, u is very small, except at the moving boundary. This is an example of a boundary layer. This boundary layer is not a typical one (more on this later), but one can see that its thickness is $\approx \nu$.

3.6 Low Reynolds Number Flow: Stokes Flow

Recall that the Navier-Stokes equations for steady flow of an incompressible fluid are

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}.$$

Now, the ratio of the viscous and inertia terms are such that

$$\frac{1}{R} = \frac{\text{viscous}}{\text{inertia}}.$$

Therefore, if $R \ll 1$, the viscous terms dominate, and we can ignore the inertia terms. Such flow is called *creeping flow*, and one has the *Stokes equations*,

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla p = \mu \nabla^2 \mathbf{u}. \tag{3.15}$$

Recall that the $\nabla^2 \mathbf{u}$-term is a viscous term, and so the second equation of the above is a balance between dynamical pressure and viscous forces.

We could also use another argument of scale out inertia terms; for example, in thin-film flows, where viscosity acts over small length scales, such as in lubrication layers. Hence, with the usual scalings, $\mathbf{u} \cdot \nabla \mathbf{u} \sim u^2/d$ and $\nu \nabla^2 \mathbf{u} \sim \nu u/h^2$, the creeping flow requirement becomes

$$R \ll 1 \quad \Rightarrow \quad \frac{u^2}{d} \ll \frac{\nu u}{h^2}.$$

See Figure (3.7). For that setup, it can be shown that there is a solution to the creeping flow if

$$u > 2.014 \frac{g h^2}{\nu},$$

where h is the mean thickness of the layer. That is, the film will remain on the rod, if this holds, otherwise the fluid will drain off.

Notice that the Stokes equations (3.15) are linear, and so solutions are reversible. That is, the flows which they describe are reversible. Stokes flow is very important in colloids and laser trapping of particles & biological fluid motion. Now, symmetry of a system – such as in the boundary conditions – also gives reversibility. For example, if a "swimmer" in a viscous fluid has a tail which flaps from side to side, then there is a symmetry, which means that the flow is reversible, which means that the swimmer will not move. If the swimmer has a screw type-tail, then the symmetry is lost, and the swimmer can move.

3.7 Stream Functions

Recall that for incompressibility, we have that

$$\nabla \cdot \mathbf{u} = 0,$$

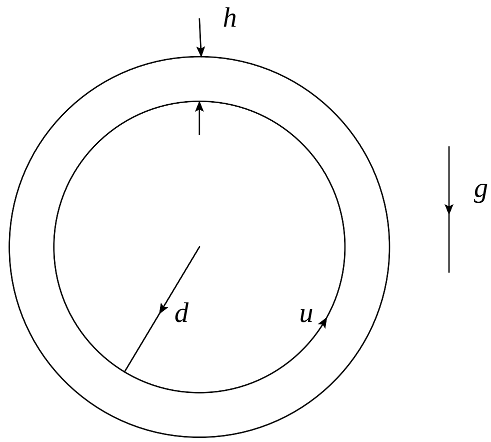


Figure 3.7: An example of a thin viscous layer between two surfaces; it can be shown that there is a solution to the creeping equations of motion.

or, for a 2D flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Suppose we introduce some function ψ, such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

then we see that the divergence of $\mathbf{u}$ is still zero. Thus, we have replaced two functions, u, v by a single *stream function* ψ.

Notice that

$$\begin{aligned} d\psi &= \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \\ &= -vdx + udy. \end{aligned} \tag{3.16}$$

Now, on a streamline,

$$\frac{dx}{u} = \frac{dy}{v} \quad \Rightarrow \quad v = u\frac{dy}{dx},$$

hence, using this in (3.16), we see that

$$d\psi = -udy + udy = 0.$$

That is, $d\psi$ is zero on a streamline. This is equivalent to saying

$$\psi = \text{const on a streamline.}$$

Also, one could say that streamlines are lines of constant ψ. Streamlines cannot cross, except at stagnation points.

3.7.1 Stokes Flow Past a Sphere

Consider a slow moving sphere in a static fluid. For example, consider a sphere of diameter 1cm, and having a speed $v = 2\text{cm s}^{-1}$. Then, the Reynolds number, $R = vd/\nu$, for various viscosities of fluids can be computed:

Fluid	**Viscosity ν at 20°C**	**Reynolds Number R**
water	1cS	200
olive oil	10cS	20
glycerine	2×10^3cS	0.1
golden syrup	10^4cS	0.02

Notice that all will be laminar flows, but only the last two will be Stokesian (i.e. have $R \ll 1$). Even if we were to reduce u by a factor of 10, only the last two would be Stokesian.

So, we need either small length scales or large viscosities ("sticky fluids") to get Stokes flow.

We shall not give a full solution to the problem, we shall only give an outline.

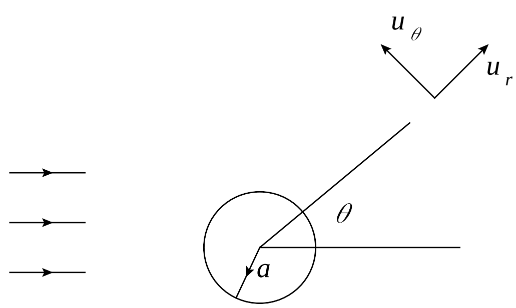


Figure 3.8: The general setup, and coordinate system, for a stationary sphere in a flow.

See Figure (3.8) for a schematic of the setup. We use spherical polar coordinates, centred on the sphere. The solution will be axially symmetric, so that

$$u_\phi = 0.$$

The continuity equation, $\nabla \cdot \mathbf{u} = 0$, in spherical polars, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) = 0.$$

The stream function ψ that satisfies this is such that

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

In Stokes' equations, we have a term $\nabla^2 \mathbf{u}$. We will use the vector identity

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u} = -\nabla \times \nabla \times \mathbf{u},$$

to clean the expression up. The second equality follows due to the incompressibility of the fluid. Hence, in spherical polars, the Stokes equations read

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{\mu}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \right], \\ \frac{1}{r} \frac{\partial p}{\partial \theta} &= -\frac{\mu}{r \sin\theta} \frac{\partial}{\partial r} \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \right]. \end{aligned}$$

We can eliminate p to find

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right) \right]^2 \psi = 0.$$

This homogeneous differential equation can be solved via standard techniques, such as a power series solution, subject to appropriate boundary conditions. The boundary conditions we use are

$$\begin{aligned} u_r(r \to \infty) &= u_0 \cos\theta, & u_\theta(r \to \infty) &= -u_0 \sin\theta, \\ u_r(r = a) &= 0, & u_\theta(r = a) &= 0. \end{aligned}$$

The first line of conditions impose a free streaming flow from infinity, with free-streaming speed u_0, and the second set impose no-slip (that is, no flow across material surface). The solution to the differential equation is then given by

$$\begin{aligned} u_r &= u_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3} \right), \\ u_\theta &= -u_0 \sin\theta \left(1 - \frac{3a}{4r^2} - \frac{a^3}{2r^3} \right), \\ p - p_0 &= -\frac{3\mu u_0 a}{2r^2} \cos\theta; \end{aligned}$$

where p_0 is the hydrostatic pressure. We can use these solutions to compute the streamfunction,

$$\psi = \frac{u_0 r^2}{\sin^2\theta} \left(\frac{3a}{4r} - \frac{a^3}{4r^3} \right).$$

Notice that the "lead term" is $1/r$, rather than $1/r^2$, which means that the influence of the flow can travel very far, meaning that boundaries can have a large effect in experiment.

So, we have computed the solution to a low Reynolds number flow past a sphere.

Notice that the solution is symmetric – this is a reflection of the linearity of the problem.

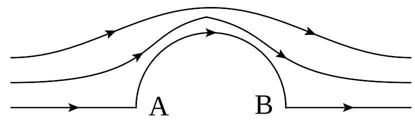


Figure 3.9: The streamlines in a Stokes flow past a sphere.

Let us then consider the point force at the surface of the sphere. The shear force is

$$\begin{aligned} \sigma &= -\mu \left(\frac{\partial u_\theta}{\partial r} \right)_{r=a} \sin\theta - (p - p_0)_{r=a} \cos\theta \\ &= \frac{3\mu u_0}{2a}. \end{aligned}$$

Notice that this is independent of θ. Hence, the total force D, is just σ multiplied by the surface area of the sphere,

$$D = 4\pi a^2 \sigma = 6\pi\mu a u_0.$$

Now, consider the balance between this total force, and a bouyancy force,

$$6\pi\mu a u_0 = \frac{4}{3}\pi a^3 \left(\bar{\rho} - \rho\right) g,$$

where ρ is the density of the fluid, and $\bar{\rho}$ the density of the sphere – this is only true for $R \ll 1$. Then, one can easily use this to write the Reynolds number,

$$R = \frac{2au}{\nu} = \frac{4a^3 g}{9\nu^2} \left(\frac{\bar{\rho}}{\rho} - 1 \right) \ll 1.$$

Consider a sand grain falling in water. Then, $\bar{\rho}/\rho \sim 2$ and $\nu = 1$cS (at $T = 20°$C), hence, the Reynolds number must be smaller than $R = 4.4 \times 10^4 a^3$. That is, for a Stokesian flow, $a < 0.03$mm. Further consider a (solid) raindrop, with $\bar{\rho}/\rho \sim 780$, $\nu = 15$cS, then $a < 0.04$mm. Finally consider a steel ball in syrup: $\bar{\rho}/\rho \sim 7, \nu = 1.2 \times 10^4$cS giving $a < 1.65$mm.

We define the *viscous drag coefficient*

$$C_{\mathrm{D}} = \frac{2D}{\rho u^2 a^2} = \frac{6\pi}{R}. \tag{3.17}$$

The symbol C_{D} is used as a general viscous drag coefficient, but here we have written an expression only for Stokesian flows; either way, C_{D} is unimportant for $R > 1$.

Now consider the case of a non-rigid sphere – an inviscid drop in an immiscible viscous fluid. For example, a water-oil interface. The normal components of the velocity are equal, and there is no relative motion of the fluids at the interface; tangential stress is continuous across the interface – equal and opposite across the interface. All other conditions are as in the solid sphere case. The solution is

$$u = \frac{a^2 g}{3\nu}\left(\frac{\bar{\rho}}{\rho} - 1\right)\frac{\mu + \bar{\mu}}{\mu + \frac{3}{2}\bar{\mu}}.$$

We have that $\mu/\bar{\mu} = 0$ corresponds to a solid, which gives the Stokes result; and $\bar{\mu}/\mu = 0$ for a viscous liquid. This result can be used in experiment to test the purity of a liquid. If a liquid has impurities – dirt – then, the impurities accumulate at the interface, giving an essentially solid shell, thus changing the speed.

3.8 Steady Inviscid Flows

Let us now consider flows which have $\nu \to 0$, so that $R \to \infty$. This also works for flows with large u_0.

The steady Navier-Stokes equations are

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0;$$

or, if we appropriately scale them, we have

$$\mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\frac{1}{\rho}\nabla^* p^* + \frac{1}{R}(\nabla^*)^2 \mathbf{u}^*.$$

Then, letting $R \to \infty$, we see that this just becomes (leaving off the stars),

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p. \tag{3.18}$$

This is known as *Euler's equation*. This is now a first order differential equation, which means that we require less boundary conditions. We tend to retain the impermeability boundary condition, but reject the no-slip condition. This gives something of a paradoxical situation; as the Reynolds number R increases, the Euler solutions are found to become better and better approximations, however, in practice, viscosity is still effective at boundaries, so that no-slip holds, but we can choose to ignore no-slip in solving Euler's equations. The resolution of this comes when we make a better choice of length scales, giving boundary layers.

Let us consider a streamline in an inviscid flow, governed by Euler's equation. Suppose we denote the distance along the streamline by ℓ. At any point on the streamline, the component of Euler's equation along the streamline is just

$$\rho q \frac{\partial q}{\partial \ell} = -\frac{\partial p}{\partial \ell},$$

where q is some speed. Integrating this easily gives us that

$$\frac{1}{2}\rho q^2 + p = \text{const on a streamline.} \tag{3.19}$$

This is *Bernoulli's equation*. One can think of the first term as a "kinetic energy". Hence, if we have high pressure, then we must have low speed; and vice-versa.

We could extend Bernoulli's equation, by considering the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u} \times \nabla \times \mathbf{u}$$

in the LHS of (3.18). Using the definition of vorticity,

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u},$$

but imposing irrotational flow (more on this later), then we see that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla q^2.$$

Therefore, using this in Euler's equation (3.18), we see that

$$\nabla \left(\frac{1}{2}\rho q^2 + p \right) = 0,$$

which only holds for irrotational flow.

Therefore, we have seen that $\frac{1}{2}\rho q^2 + p$ is a constant throughout the flow, using Bernoulli, where the constant is the same per streamline.

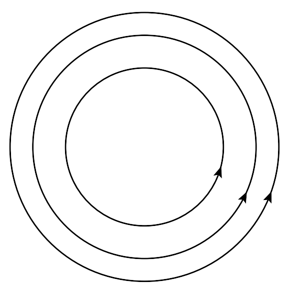


Figure 3.10: Circular streamlines in a uniform flow. The constant of Bernoulli is the same along a given streamline.

3.8.1 D'Alembert's Paradox

A consequence of the inviscid flow model, is the lack of drag for a symmetric object placed in a flow.

Consider a cylinder placed in a flow, as in Figure (3.9). Consider the streamline that hits the middle of the cylinder (such as at point A), and that it travels around and leaves the cylinder at B. Then, by Bernoulli,

$$p_{\mathrm{A}} = p_{\mathrm{B}} = \frac{1}{2}\rho u_0^2 + p_0.$$

That is, the pressure difference is the same along the streamline, which means that there is no net pressure difference, which means that there is no drag. Therefore, upon placing such a cylinder in such a flow, one would not see the cylinder get carried along by the flow. This pressure distribution is the same as for the undisturbed flow (i.e. flow with no cylinder).

Infact, it can be shown that there will be zero drag for non-symmetric obstacles.

3.8.2 Applications of Bernoulli's Theorem

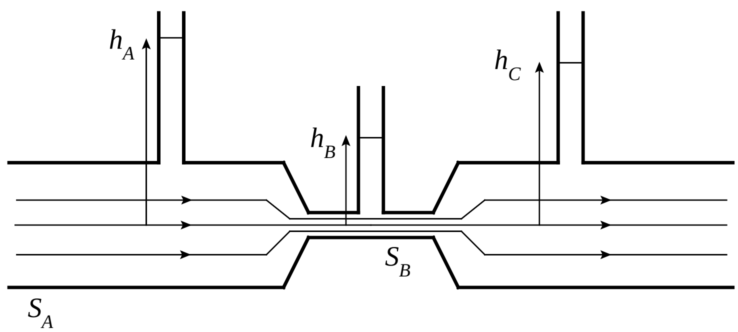


Figure 3.11: The Venturi tube for measuring flow rates. The cross-sectional areas are denoted S_{A} and S_{B}.

The Venturi Tube Consider the setup in Figure (3.11). We have a flow of density ρ coming in from the right, through a tube of cross-sectional area S_{A}, the flow is forced into a smaller tube of cross-section S_{B}, and comes out again the other side. At each tube is a manometer tube, which allows the height of the fluid to be read off.

Notice that streamlines are closer together in the middle tube, than in the LHS tube – this means that the flow is faster in the middle than the LHS tube. The cross-sectional area of the RHS tube is the same as the LHS tube, so, one would naively assume that $h_{\mathrm{A}} = h_{\mathrm{C}}$ – this is not true, as vortices tend to be created when the fluid leaves the smaller tube, thus leaving $h_{\mathrm{C}} < h_{\mathrm{A}}$. The Venturi tube is used to measure fluid flow rates, as one can compute the flow rate given h_{A} and h_{B}, as we shall now show.

Consider applying Bernoulli's equation (3.19) to the central streamline. Then,

$$\begin{aligned} \frac{1}{2}u_{\mathrm{A}}^2 + \frac{p_{\mathrm{A}}}{\rho} &= \frac{1}{2}u_{\mathrm{B}}^2 + \frac{p_{\mathrm{B}}}{\rho}, \\ \Rightarrow \quad u_{\mathrm{A}}^2 - u_{\mathrm{B}}^2 &= \frac{2}{\rho}\left(p_{\mathrm{B}} - p_{\mathrm{A}}\right). \end{aligned}$$

Now, the pressure difference is just

$$p_{\mathrm{A}} - p_{\mathrm{B}} = \rho g \left(h_{\mathrm{A}} - h_{\mathrm{B}}\right),$$

and hence

$$u_{\mathrm{A}}^2 - u_{\mathrm{B}}^2 = -2g\left(h_{\mathrm{A}} - h_{\mathrm{B}}\right). \tag{3.20}$$

Now, the equation of continuity, for an incompressible fluid,

$$u_{\mathrm{A}} S_{\mathrm{A}} = u_{\mathrm{B}} S_{\mathrm{B}},$$

allows us to see that

$$u_{\mathrm{B}} = \frac{u_{\mathrm{A}} S_{\mathrm{A}}}{S_{\mathrm{B}}},$$

and hence upon substitution into (3.20),

$$u_{\mathrm{A}} = \left(\frac{2g(h_{\mathrm{A}} - h_{\mathrm{B}})}{\frac{S_{\mathrm{A}}^2}{S_{\mathrm{B}}^2} - 1}\right)^{1/2}.$$

Therefore, we have a way of measuring the input flow speed for a fluid entering a Venturi tube.

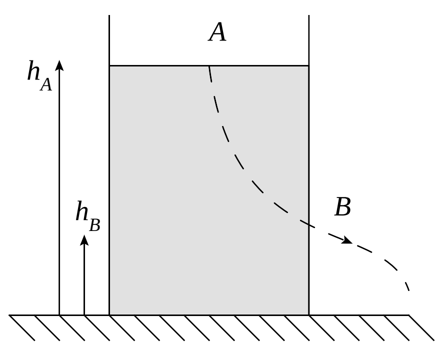


Figure 3.12: Flow from a hole in a tank. The fluid height is h_{A}, and the hole is at a height h_{B}, and the fluid leaves the hole with velocity u.

Flow from a Hole in a Tank Consider a setup as in Figure (3.12). Let us assume that the top level of the fluid drops slowly, so that we can neglect the velocity of the free surface. Hence, $u_{\mathrm{A}} = 0$. Thus, applying Bernoulli's equation to the streamline shown in the diagram,

$$\frac{p_{\mathrm{A}}}{\rho} = \frac{p_{\mathrm{B}}}{\rho} + \frac{1}{2}u^2. \tag{3.21}$$

The pressure difference is just

$$p_{\mathrm{A}} - p_{\mathrm{B}} = \rho g\left(h_{\mathrm{A}} - h_{\mathrm{B}}\right) = \rho g h,$$

where $h \equiv h_{\mathrm{A}} - h_{\mathrm{B}}$ is the distance between the hole and the top of the fluid. Hence, using this in (3.21), we see that

$$\frac{1}{2}u^2 = gh,$$

and hence the speed of the fluid when it leaves the hole

$$u = \sqrt{2gh}.$$

4 Vorticity

Recall that we defined the curl of the velocity field to be *vorticity*,

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}. \tag{4.1}$$

Now, consider a 2D flow,

$$\mathbf{u} = (u(x,y,t), v(x,y,t), 0)\,,$$

hence, one can easily see that vorticity has only one component,

$$\boldsymbol{\omega} = (0,0,\omega)\,, \quad \omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Vorticity gives a local measure of spin of a fluid element – it does not tell you that the global velocity field is rotating.

Consider that

$$v(x+\delta x,y,t) - v(x,y,t) = \frac{\partial v}{\partial x}\delta x, \quad u(x,y+\delta y,t) - u(x,y,t) = \frac{\partial u}{\partial y}\delta y;$$

which follows from the definition of partial derivatives. Hence, taking their difference, we have

$$\frac{1}{2}\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Thus, vorticity is the average angular velocity of two short fluid elements.

Consider the vector identity,

$$\nabla \times (\mathbf{u}\cdot\nabla\mathbf{u}) = \mathbf{u}\cdot\nabla(\nabla\times\mathbf{u}) - (\nabla\times\mathbf{u})\cdot\nabla\mathbf{u} + (\nabla\cdot\mathbf{u})(\nabla\times\mathbf{u}).$$

The last term is zero by incompressibility, $\nabla\cdot\mathbf{u} = 0$, and we write the other terms in terms of $\boldsymbol{\omega}$, so that

$$\nabla \times (\mathbf{u}\cdot\nabla\mathbf{u}) = \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u}. \tag{4.2}$$

Let us write the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F},$$

where $\mathbf{F}$ is some body force; take its curl,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times\mathbf{u}\cdot\nabla\mathbf{u} = \nu\nabla^2\boldsymbol{\omega} + \nabla\times\mathbf{F}.$$

Now, as we can write $\mathbf{F}$ as the grad of a scalar, and the curl grad is zero, then the curl of $\mathbf{F}$ is zero. Also, we write the second term on the LHS using (4.2),

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \nu\nabla^2\boldsymbol{\omega},$$

we can then write the first two terms on the LHS in terms of the material derivative, so that

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}. \tag{4.3}$$

This is known as the *vorticity equation.*

We commonly will consider steady flows, so that

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = 0.$$

What this means, is that if there is no initial vorticity, then there is never vorticity.

A flow is called *irrotational* if $\boldsymbol{\omega} = 0$.

4.1 Examples

Let us consider some examples.

4.1.1 Irrotational Flow

Consider a flow with

$$u_\theta = \frac{\Omega a^2}{r}, \quad u_r = u_z = 0.$$

Then, vorticity is given by

$$\boldsymbol{\omega} = \nabla_\theta \times \frac{\Omega a^2}{r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Omega a^2}{r}\right) = 0.$$

Now, this is only defined for $r \neq 0$; so that we can only say that the flow is irrotational off-axis. To see this, consider Stokes' theorem,

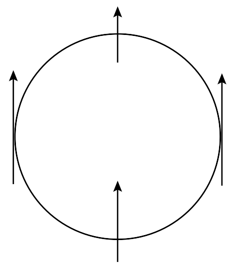
$$\oint \mathbf{u}\cdot d\boldsymbol{\ell} = \int \nabla\times\mathbf{u}\cdot d\mathbf{S} = \int \boldsymbol{\omega}\cdot d\mathbf{S}.$$

So, integrating at fixed r gives $2\pi\Omega a^2$. However, we showed that the flow was irrotational. Therefore, everywhere, except at $r = 0$, the flow is irrotational; and all vorticity is concentrated at $r = 0$. See Figure (4.1)a.

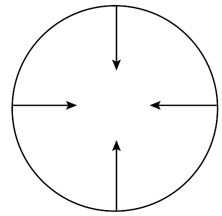
4.1.2 Solid Body Rotation

Consider a fluid element as in Figure (4.1)b, where $\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{r}$, so that vorticity is given by

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{\Omega}\times\mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 2\Omega\hat{\mathbf{k}}.$$



(a) Irrotational flow.



(b) Solid body rotation.

Figure 4.1: Irrotational flow – as the fluid moves round, it retains its alignment. Also shown is solid body rotation.

Hence, we see that a change in the orientation of a fluid element gives rise to vorticity; where vorticity is twice the rotational velocity of the fluid element.

4.1.3 Hele-Shaw Flow

Consider the setup as in Figure (4.2). We are considering thin film flow, where two glass plates have a very narrow uniform gap h (of the order mm).

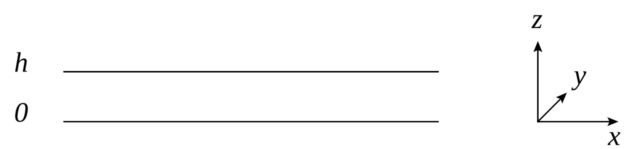


Figure 4.2: Hele-Shaw flow between two glass plates which are very close together.

We impose the no-slip boundary condition at $z = 0, h$. In the z-direction, viscosity dominates, so that the Navier-Stokes equations become

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h - z), \quad v = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h - z).$$

Notice that this is quadratic in z. Also notice that each flow component u, v depend upon z, but, their ratio does not. Hence, the pressure p is a function of x, y only. If we cross-differentiate, and subtract, we arrive at the interesting result

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

This is the expression for vorticity, and hence Hele-Shaw flow is equivalent to an irrotational flow. Applications of such flow include flow through porous media – such as soils and oil bearing rock – which can be considered statistically uniform.

4.2 Irrotational Flow & The Complex Potential

Irrotational flow is important in many practical situations, including flow far from boundaries and atmospheric flows.

Let us consider only 2D irrotational flows, so that

$$\nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

Now, we can replace the velocity vector by a *velocity potential* ϕ, so that

$$\mathbf{u} = -\nabla \phi.$$

Hence,

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}.$$

The incompressibility statement $\nabla \cdot \mathbf{u} = 0$ allows us to also write

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x};$$

where ψ is the streamfunction, as already discussed. Hence, equating these expressions, we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{4.4}$$

These equations are known as the *Cauchy-Riemann equations*, and are familiar from complex analysis. Notice that both the streamfunction ψ and velocity potential ϕ are harmonic functions;

$$\nabla^2 \phi = \nabla^2 \psi = 0.$$

Also notice that

$$\nabla \phi \cdot \nabla \psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0,$$

so that we can see that equipotential lines are orthogonal to streamlines – except at stagnation points, where velocity is zero.

If the partial derivative are zero, we can define the *complex potential* w,

$$w = \phi + i\psi. \tag{4.5}$$

Thus, we see that the complex potential is a function of a complex variable $z = x + iy$. Expressing flows in terms of the complex potential w makes analysis of flows much simpler. Let us consider some examples.

4.2.1 Examples of w-flows

Consider a flow with
$$w = z^2.$$
Then,
$$z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2ixy.$$
Therefore, as ϕ is the real part of w, and ψ the complex part,
$$\phi = x^2 - y^2, \quad \psi = 2xy.$$

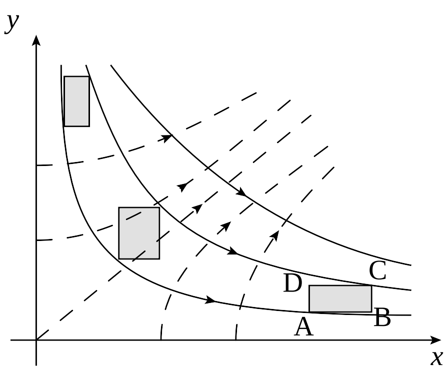


Figure 4.3: Flow round a corner, given by $w = z^2$. Solid lines are streamlines, dashed lines are lines of equipotential.

See Figure (4.3) for a visualisation of the flow. Notice that the box $ABCD$ retains its area, but as it moves from right to left, AB, DC are compressed, and AD, BC are stretched. The box changes shape, but keeps its sides parallel to the axes – hence, the flow is irrotational.

A second case we consider is a generalisation of $w = z^2$;
$$w = Ua\left(\frac{z}{a}\right)^{\pi/\alpha}, \tag{4.6}$$
so that flow motion takes place contained by two boundaries with internal angle α, a is a length scale and U the speed of the flow.

With reference to Figure (4.4), we see various flow diagrams for various angles α.

Also, consider a flow with complex potential
$$w = \frac{Uz^2}{a}. \tag{4.7}$$
Hence, streamlines are given by $\psi = \frac{2U}{a}xy$

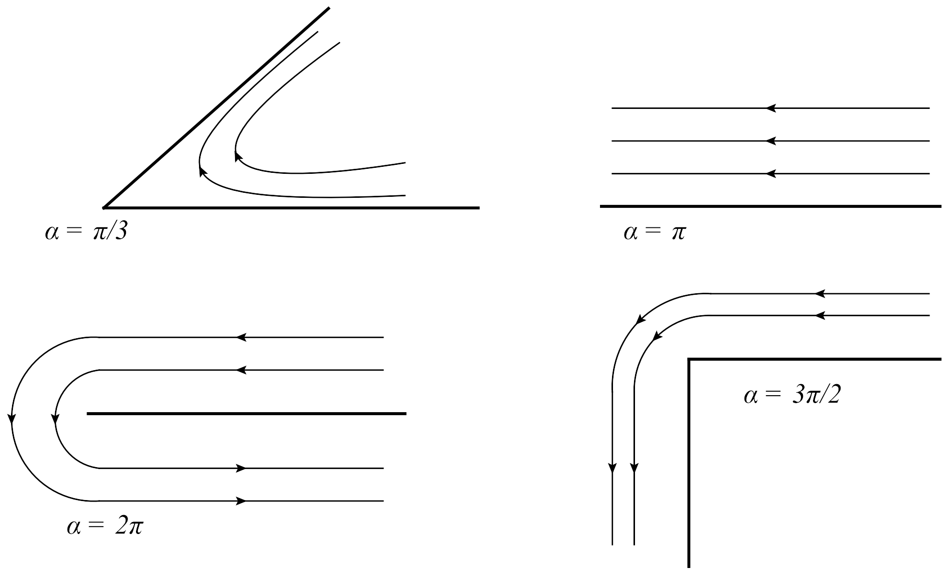


Figure 4.4: Flows (4.6) with various angles α.

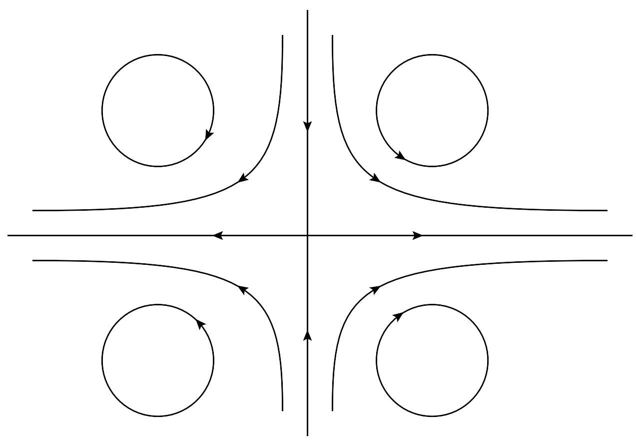


Figure 4.5: Stagnation point flow (4.7).

See Figure (4.5) for the streamlines – the circles have been added to show how such a flow could be used to stretch polymers, the so called 4-roll mill.

Finally, consider a flow with

$$w = \frac{Ua^2}{z}.$$

Then, the streamlines are given by

$$\psi = -\frac{Ua^2}{r}\sin\theta = -\frac{Ua^2 y}{x^2 + y^2}.$$

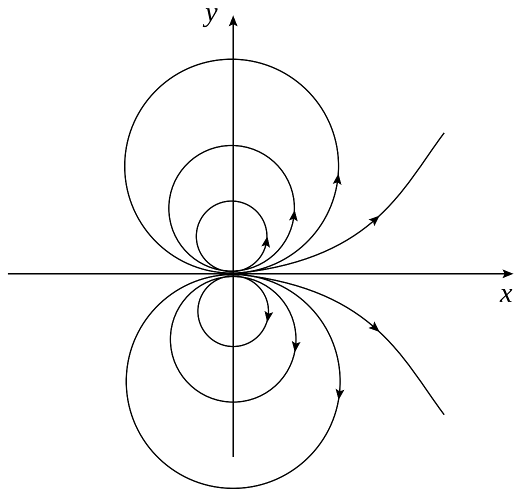


Figure 4.6: Doublet point flow.

With reference to Figure (4.6), we see that the streamlines are circles; they all touch the x-axis at the origin. The motion is the so-called doublet, with a combined source and sink at the origin.

We have seen that in the special case of irrotational solenoidal flow, the equations of motion are linear – in general, inviscid flows are governed by the non-linear Euler equations. The streamlines ψ exist for rotational and irrotational flows, but the velocity potential ϕ only exists for irrotational flows. Some flows have regions of irrotational flow separated by rotational parts.

5 Lubrication Theory

Lubrication theory is concerned with thin film flows; where a sensible choice of length scales allow a simplification of the Navier-Stokes equations, which allows various predictions to be made.

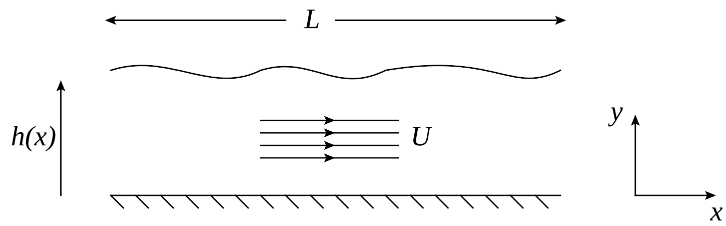


Figure 5.1: A schematic of the thin film flow, and the scales involved. We have a steady flow between two plates; the bottom is flat, and the top varies as one travels along. We have that $h \ll L$.

With reference to Figure (5.1), we see a schematic of some flow between two plates; the flow field is the thin film. We have that U is some characteristic velocity scale of the flow in the x-direction, L the length of the flow, and h the height scale of the flow. We shall impose no-slip boundary conditions at the surfaces,

$$u(y=0) = u(y=h) = 0.$$

As usual, we take the velocity vector to have components $\mathbf{u} = (u, v)$. We now want to make order-of-magnitude estimates, in terms of these scales; how big certain terms in the NS equations will be.

The velocity u will vary by $\mathcal{O}(U)$ in a distance y by an order $\mathcal{O}(h)$ (this is saying that the film is thin, and that there is not much variation "up" the flow). Therefore,

$$\frac{\partial u}{\partial y} \sim \mathcal{O}\left(\frac{U}{h}\right).$$

Similarly, the variation in u along the x-direction will be $\mathcal{O}(L)$, so that

$$\frac{\partial u}{\partial x} \sim \mathcal{O}\left(\frac{U}{L}\right). \tag{5.1}$$

And hence,

$$\frac{\partial^2 u}{\partial y^2} \sim \mathcal{O}\left(\frac{U}{h^2}\right), \quad \frac{\partial^2 u}{\partial x^2} \sim \mathcal{O}\left(\frac{U}{L^2}\right). \tag{5.2}$$

Now, recall that the viscous and inertial terms in the NS equations entered via

$$\nu \nabla^2 \mathbf{u} \sim \mathcal{O}\left(\frac{\nu U}{h^2}\right) \qquad \Rightarrow \qquad \text{viscous,}$$
$$\mathbf{u} \cdot \nabla \mathbf{u} \sim \mathcal{O}\left(\frac{U^2}{L}\right) \qquad \Rightarrow \qquad \text{inertia,}$$

after using (5.1) and (5.2); and hence in a comparison of inertia to viscous forces,

$$\frac{\text{inertia}}{\text{viscous}} \sim \frac{UL}{\nu}\left(\frac{h}{L}\right)^2.$$

We can now notice that the first factor is the Reynolds number R, so that

$$\frac{\text{inertia}}{\text{viscous}} \sim R\left(\frac{h}{L}\right)^2. \tag{5.3}$$

Recall that we could linearise the NS equations into the Stokes equations, in the limit that $R \ll 1$. This amounted to requiring viscous terms dominating over the inertia terms; and we neglected the inertia terms. We now see that we can neglect the inertia terms in favour of the viscous terms, even if R is large, if h is very small (relative to L).

A common example of such a situation is when a sheet of paper slides over a surface. Taking the speed scale to be $U = 100\text{mm s}^{-1}$, the length of the sheet of paper $L = 300\text{mm}$, the height of the paper above the surface over which it it sliding, $h = 0.5\text{mm}$ and the viscosity of air $\nu = 15\text{mm}^2\text{s}^{-1}$. Therefore, using these numbers,

$$R = \frac{UL}{\nu} = 2000, \quad R\left(\frac{h}{L}\right)^2 = 0.05 \ll 1.$$

Therefore, we say that such a system is in the *lubrication limit.*

5.1 The Lubrication Equations

We shall now derive the lubrication equations, by considering the length scales in the NS equations, and noting which terms are negligible relative to others.

We shall now introduce some length and velocity scales,

$$x = L\tilde{x}, \quad y = h\tilde{y}, \quad u = U\tilde{u}, \quad v = V\tilde{v},$$

where all tilde-quantitities have the same scale, and actual scales are set by quantities such as U, L. Upon substitution of the scale into the incompressibility condition,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

we easily find that

$$\frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{h}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0.$$

The non-trivial solution of this only occurs if the pre-factors are of the same order,

$$\frac{U}{L} \sim \frac{V}{h} \quad \Rightarrow \quad V \sim \frac{hU}{L}.$$

Now let us scale the x-component of the NS equation,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Hence, putting in our scales,

$$\frac{U^2\tilde{u}}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{hU^2\tilde{v}}{Lh}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{P}{\rho L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu U}{h^2}\left(\frac{h^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right),$$

where we have introduced the pressure scale $p = P\tilde{p}$. Now, if $h \ll L$, then $h^2/L^2 \ll 1$, so that the second term on the RHS is negligible. Therefore, also dividing by the factor $\nu U/h^2$,

$$\frac{1}{\frac{\nu U}{h^2}}\frac{U^2}{L}\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) = -\frac{P}{\rho L}\frac{1}{\frac{\nu U}{h^2}}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2},$$

rearranging slightly, easily gives

$$\left(\frac{hU}{\nu}\right)\frac{h}{L}\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}} = -\frac{Ph^2}{\rho\nu LU}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}.$$

Now, in the lubrication limit, the prefactor on the LHS is negligible (also, we stated previously that we want to neglect the inertia terms, which we have been able to show that we are allowed to do); also, the prefactor to the far RHS term is $\mathcal{O}(1)$. Therefore, to balance, we require

$$\frac{Ph^2}{\rho\nu LU} \sim \mathcal{O}(1)$$

to balance this equation. This gives us a pressure scale P.

5.2 Slider Bearing

With reference to Figure (5.2), we see a schematic of the setup for a slider bearing.

Now, as we have a small angle, one can see that

$$\alpha = \frac{d}{L}, \quad \alpha \ll 1;$$

which means that

$$\frac{\text{inertial}}{\text{viscous}} \sim R\alpha^2.$$

We can consider an example; such as an oil-bearing, with numbers

$$U_0 = 1.2\text{ms}^{-1}, \quad L \sim 100\text{mm}, \quad d \sim 0.2\text{mm}, \quad \nu \sim 37\text{mm}^2\text{s}^{-1};$$

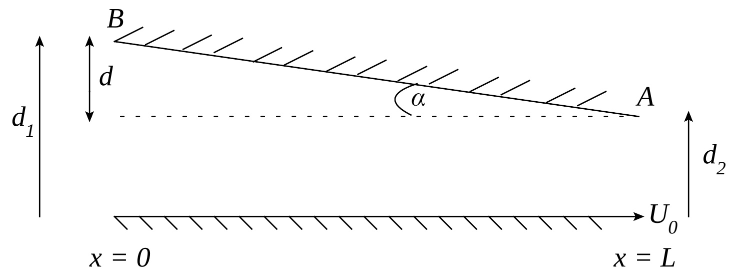


Figure 5.2: A slider bearing. Notice that the top plate makes a small angle α with the bottom plate; and that the lower plate moves with speed U_0.

so that we have

$$R \sim 30,000, \quad \alpha = \frac{d}{L} = 2 \times 10^{-1} \quad \Rightarrow \quad R\alpha^2 \sim 0.1;$$

and therefore, such a system is in the lubrication limit.

Recall that we derived that the NS equations, in the lubrication limit, reduced to

$$-\frac{1}{\mu}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5.4}$$

Infact, if we had done the same analysis for the y-direction, we would have found that $\frac{\partial p}{\partial y} = 0$, which means that $p = p(x)$ only. Thus, we see that we have a flow driven by a pressure gradient and boundary conditions. We can solve this equation by selecting a particular x, where, say, the film thickness is d. Therefore, we can easily integrate to find

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} y(d-y) + U_0\left(\frac{d-y}{d}\right).$$

We can first note that the first term is just a pressure driven Poiseulle flow, and that the second term is a plane Couette flow. We can secondly check that our boundary conditions are satisfied in this solution (i.e. $u(0) = U_0$ and $u(d) = 0$).

Using this velocity profile, we can compute various quantities. For example, the volume flux per unit width of layer is given by

$$Q = \int_0^d u\, dy = -\frac{1}{12\mu}\frac{\partial p}{\partial x} d^3 + \frac{U_0 d}{2}.$$

Now, this must be independent of x, as the volume flux in must be equal to the volume flux out (by continuity). Hence, we can use Q to compute $\frac{\partial p}{\partial x}$. Rearranging this easily provides

$$\frac{\partial p}{\partial x} = 6\mu\left(\frac{U_0}{d^2} - \frac{2Q}{d^3}\right),$$

noting that $d \equiv d(x)$. So, we can integrate this, using the chain rule,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial d}\frac{\partial d}{\partial x},$$

where $d(x) = d_1 - \alpha x$, so that

$$\frac{\partial p}{\partial d} = -\frac{6\mu}{\alpha}\left(\frac{U_0}{d^2} - \frac{2Q}{d^3}\right).$$

Hence, integrating

$$p(d) = \frac{6\mu}{\alpha}\left(\frac{U_0}{d} - \frac{Q}{d^2}\right) + C,$$

where C is a constant of integration. We can find C by imposing that at $x = 0$, we have a pressure $p = p_0$, and hence

$$p - p_0 = \frac{6\mu}{\alpha}\left[U_0\left(\frac{1}{d} - \frac{1}{d_1}\right) - Q\left(\frac{1}{d^2} - \frac{1}{d_1^2}\right)\right].$$

We could consider the case where the block is completely immersed in the fluid (for example, the paper is completely immersed in air). Then, the pressures at either end are the same, so that $p(d = d_2) = p_0$. Hence,

$$Q = U_0\frac{d_1 d_2}{d_1 + d_2}, \tag{5.5}$$

$$p - p_0 = \frac{6\mu U_0}{\alpha}\left(\frac{(d_1 - d)(d - d_2)}{d^2(d_1 + d_2)}\right). \tag{5.6}$$

Although we shall not go through the details, one can compute further quantities;

- Total normal force (i.e. pressure generated by the flow):

$$\int_0^L p - p_0\, dx = \frac{6\mu U_0}{\alpha^2}\left[\ln\frac{d_1}{d_2} - 2\left(\frac{d_1 - d_2}{d_1 + d_2}\right)\right].$$

- Total tangential force exerted by the fluid on the lower plane:

$$\int_0^L \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} dx = \frac{2\mu U_0}{\alpha}\left[3\left(\frac{d_1 - d_2}{d_1 + d_2}\right) - 2\ln\frac{d_1}{d_2}\right].$$

- Total tangential force exerted by the fluid on the upper plane:

$$-\int_0^L \mu\left.\frac{\partial u}{\partial y}\right|_{y=d} dx = \frac{2\mu U_0}{\alpha}\left[3\left(\frac{d_1 - d_2}{d_1 + d_2}\right) - \ln\frac{d_1}{d_2}\right].$$

The coefficient of friction is given by

$$\text{coeff of friction} = \frac{\text{tangential force on block}}{\text{normal force on block}} = \alpha f\left(\frac{d_1}{d_2}\right),$$

where $f(x)$ is some function. Now, if $\frac{d_1 - d_2}{d_1} \sim 1$, then the coefficient of friction is $\sim d_1/L$, which is very small. Therefore, the bearing has very low friction.

5.2.1 Cavitation

Consider that the pressure in an accelerated fluid is given by

$$p = p_0 + \rho(g - \beta)h.$$

Now, notice that $p < p_0$ occurs when $\beta > g$. That is, a negative pressure can occur if the acceleration is greater than that of gravity.

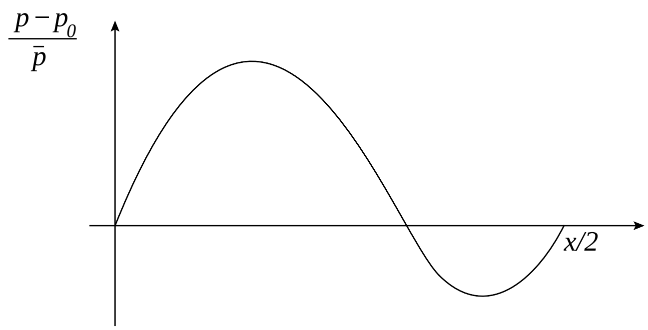


Figure 5.3: The pressure profile under a thin film flow. Notice that there is a period of negative pressure; this corresponds to where the fluid vaporises and cavitation occurs.

With reference to Figure (5.3), we see the pressure profile under a thin film flow. Basically, what this negative pressure means, is that if the pressure difference drops below the vapour pressure of the fluid, then the fluid becomes a vapour. If this occurs in systems, then the wear-and-tear on such systems can be a very large effect.

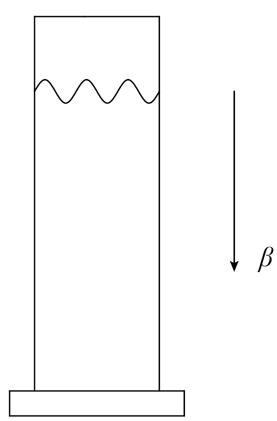


Figure 5.4: A setup of how to generate cavitation in a fluid. Consider reducing the pressure above the fluid, to something quite close to the fluids vapour pressure. Then, if the tube is accelerated downwards with acceleration $\beta > g$, the fluid will vaporise. When the tube is stopped, the vapour bubble collapses, making a loud noise and grinds away at the (reinforced) base.

With reference to Figure (5.4), we see a setup of how to generate cavitation. The pressure difference $p - p_0$ needed to allow water to cavitate, is $10^5 \mathrm{Nm}^{-2}$. Thus, taking $h = 0.42$m below the surface, an acceleration of $\beta = 1.4g$ is needed.

5.2.2 Adhesion

With reference to Figure (5.5), we see a schematic of the attempt to pull apart two plates with a thin film between them.

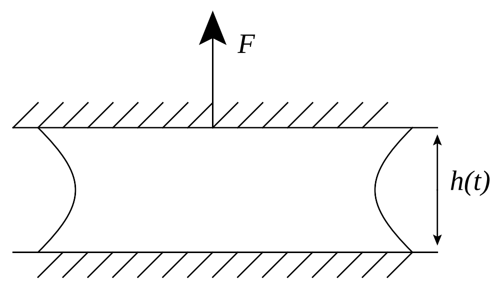


Figure 5.5: Trying to pull apart two plates, normal to the plates, is very difficult; this is the problem of adhesion.

The upwards force exerted by the fluid is

$$F_{\mathrm{up}} = -\frac{3\pi}{2}\frac{\mu a^4}{h^3}\frac{dh}{dt},$$

where a is the radius of the film. Thus, it takes a large force to pull the plates apart.

6 Aerofoil Theory

Consider a 2D aerofoil, as in Figure (6.1).

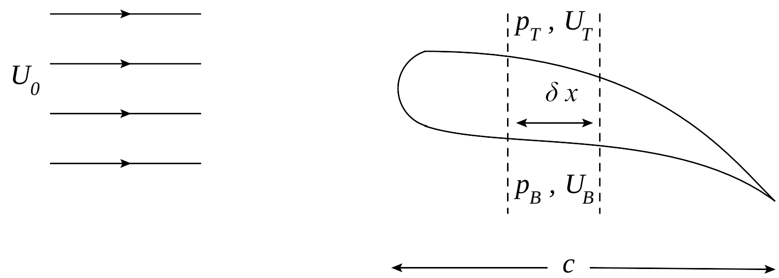


Figure 6.1: An aerofoil in a flow.

We have an aerofoil with chord length c, in a flow of speed U_0 and density ρ. The top of the wing has pressure and speed $p_{\mathrm{T}}, U_{\mathrm{T}}$, respectively. Similarly, the bottom of the wing has $p_{\mathrm{B}}, U_{\mathrm{B}}$. Notice that there is no drag on the wing, as the aerofoil is in a steady uniform irrotational flow.

The upward force per unit length on the elemental section is just

$$f = (p_{\mathrm{B}} - p_{\mathrm{T}})\, dx.$$

Now, by Bernoulli, we have that

$$p_{\mathrm{B}} - p_{\mathrm{T}} = \frac{1}{2}\rho\left(U_{\mathrm{T}}^2 - U_{\mathrm{B}}^2\right).$$

We can then write the RHS of this as an expansion,

$$\left(U_{\mathrm{T}}^2 - U_{\mathrm{B}}^2\right) = (U_{\mathrm{T}} - U_{\mathrm{B}})(U_{\mathrm{T}} + U_{\mathrm{B}}),$$

and hence,

$$p_{\mathrm{B}} - p_{\mathrm{T}} = \frac{1}{2}\rho\,(U_{\mathrm{T}} - U_{\mathrm{B}})(U_{\mathrm{T}} + U_{\mathrm{B}}).$$

We now assume that the aerofoil is thin, so that

$$U_{\mathrm{T}} + U_{\mathrm{B}} \approx 2U_0,$$

and hence,

$$p_{\mathrm{B}} - p_{\mathrm{T}} = \rho U_0\,(U_{\mathrm{T}} - U_{\mathrm{B}}).$$

Hence, the total lift per unit span is

$$L = \int_0^c f\, dx = \rho U_0 \int_0^c U_{\mathrm{T}} - U_{\mathrm{B}}\, dx. \tag{6.1}$$

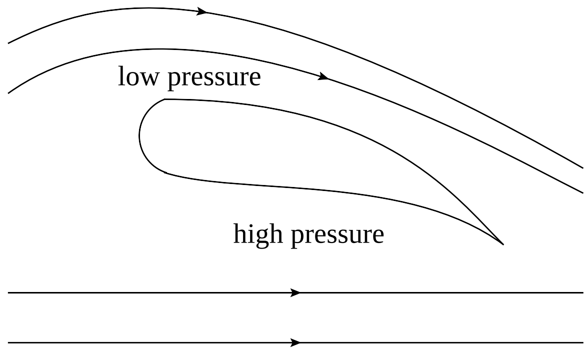


Figure 6.2: An aerofoil in a flow, and the streamlines.

With reference to Figure (6.2), we see an exaggerated schematic of the streamlines around an aerofoil. Notice that above the wing, streamlines are close together, which means high speed and low pressure. Equivalently, below the wing, streamlines are further apart, which means low speed and high pressure. Therefore, there is a lift force perpendicular to the direction of the flow.

6.1 Circulation

Consider defining

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x}, \tag{6.2}$$

where C is some closed path in a fluid. Then, Γ is called the *circulation*. We can show that an irrotational flow has zero circulation, by considering Stokes' theorem,

$$\oint_C \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S},$$

but $\nabla \times \mathbf{u} = 0$ for an irrotational flow. And hence, the circulation $\Gamma = 0$ for irrotational flow.

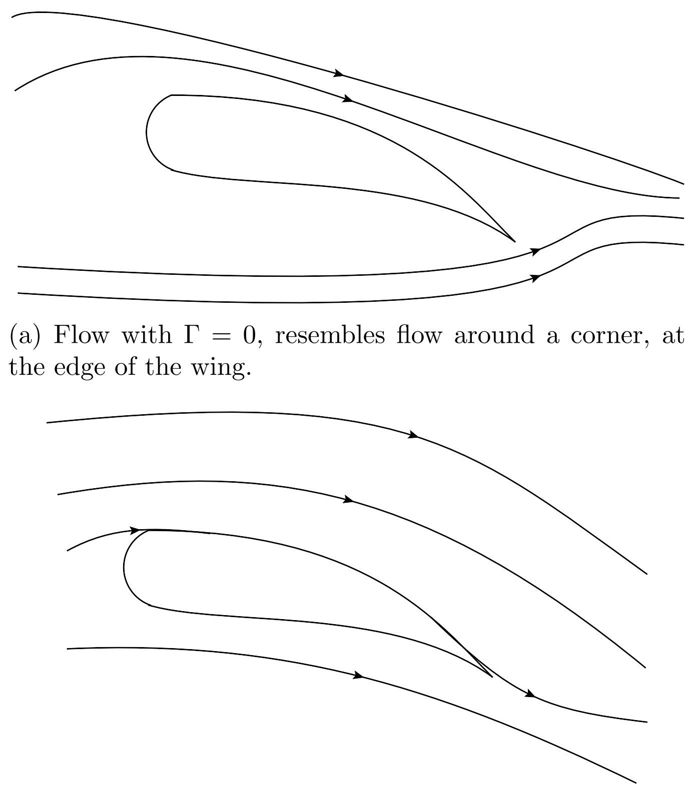
Consider a closed path around the wing, so that

$$\Gamma = \int_0^c U_{\mathrm{B}} dx + \int_c^0 U_{\mathrm{T}} dx = -\int_0^c \left(U_{\mathrm{T}} - U_{\mathrm{B}}\right) dx,$$

which is non-zero. Hence, upon comparison with our expression for lift, (6.1), we see that

$$L = -\rho U_0 \Gamma. \tag{6.3}$$

Therefore, we see that if we have lift, then we also therefore have circulation.



(a) Flow with $\Gamma = 0$, resembles flow around a corner, at the edge of the wing.

(b) Flow with $\Gamma \neq 0$, where there is a sharp separation of the flow lines from the wing.

Figure 6.3: An aerofoil in a flow, and the streamlines.

The *Joukowsi theorem* states that any 2D object in an inviscid flow links lift and circulation in this way.

See Figure (6.3)a for a flow around an aerofoil for $\Gamma = 0$, which resembles a flow around a corner, and requires a singular flow at the trailing edge. Figure (6.3)b is a flow with $\Gamma = -L/\rho U_0$, as we computed above. Viscosity helps in this case.

An interesting consequence of this, is *Kelvin's circulation theorm* which states that for any fluid governed by Euler's equations, the circulation about a material loop is conserved;

$$\frac{D}{Dt}\oint \mathbf{u}\cdot d\boldsymbol{\ell} = 0. \tag{6.4}$$

Therefore, we have seen that lift implies a finite circulation, and that circulation is conserved. Hence, in order that circulation is conserved, an eddy of opposite circulation must be created when lift is initiated (and when lift is stopped) – this is a starting vortex. This vortex is created as a consequence of the fluid being primarily an inviscid flow field – i.e. persists for a very long time.

6.2 The Magnus Effect

With reference to Figure (6.4), we see a schematic of the Magnus effect.

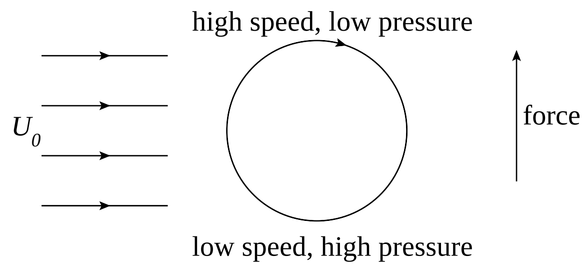


Figure 6.4: Schematic of the Magnus effect.

Consider a rotating cylinder, in a uniform steady flow – much as in the case of an aerofoil. Because the cylinder is rotating there is a pressure difference between its upper and lower sides, and therefore gives rise to lift.

This effect was actually proposed (and implemented) as a propulsion system for ships. Effectively, they secured large cylinders vertically from the deck, and rotated them. The motion of the air past the rotating cylinders produced a force which propelled the cylinders forwards, carrying the ship with it. In practice, this is hard to do, as a very large cylinder must be made to rotate, and the ship must be able to withstand some resonance effects – even so, at least two ships were made, using the Magnus effect as a propulsion system.

This would not work for purely inviscid flows, as the no-slip boundary condition fails. A similar effect was found for rotating spheres – known as the Robins effect – but the effect is more complicated to understand.

In practice, both the Magnus and Robins effects depend on the details of boundary layers; such as its roughness (this is utilised in making a footballs or cricket balls spin).

7 Boundary Layers

Eulers equations are a good approximation far from boundaries, as the Reynolds number R gets very large. We rejected the no-slip boundary condition completely – with no gradual reduction as R increases. Now, experiment shows that no-slip is always important at boundaries. The resolution of this required a new approach – the idea of a viscous boundary layer. Exactly as in the thin-film approach, we must carefully choose length scales.

A boundary layer is a region of concentrated vorticity which is long and thin. Vorticity is diffused outwards slowly via the $\nu\nabla^2\boldsymbol{\omega}$-term, whilst being convected along by the $\mathbf{u}\cdot\nabla\boldsymbol{\omega}$-term, in the vorticity equation,

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}. \tag{7.1}$$

Boundaries are generators of vorticity, being regions of strong gradients of vorticity.

So, let us see where the consideration of scales gets us. Let L be the length scale of a body (such as the diameter of a cylinder, or length of a plate), and let δ be the viscous length scale – the boundary layer thickness. Hence, in the NS equations, we see that we can write the scales of the convective (i.e. inertia) and viscous terms;

$$\mathbf{u}\cdot\nabla\mathbf{u} \sim \mathcal{O}\left(\frac{U^2}{L}\right), \quad \nu\nabla^2\mathbf{u} \sim \mathcal{O}\left(\frac{\nu U}{\delta^2}\right).$$

Thus, the inertia and viscous terms will be of comparable order when

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2},$$

which easily rearranges into

$$\frac{\delta}{L} \sim \left(\frac{UL}{\nu}\right)^{-1/2} = \frac{1}{\sqrt{R}}.$$

Hence, the ratio of the boundary layer thickness to the length scale of the body is related to the Reynolds number of the flow,

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{R}}. \tag{7.2}$$

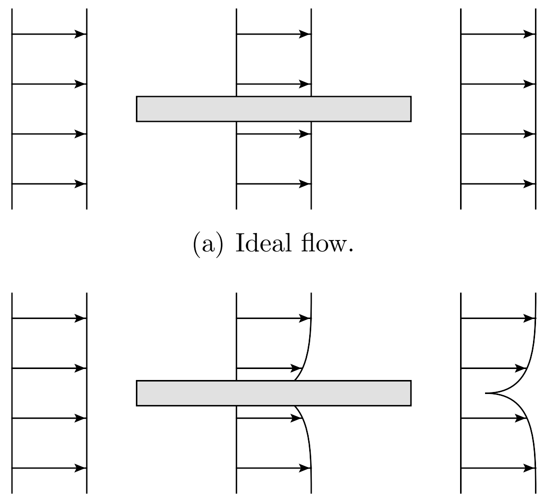
For example, considering a 20mm diameter cylinder in a flow with $R = 10,000$, there will be a boundary layer of thickness $\delta \sim 0.2$mm.

The outer flow – i.e. far from the boundary – either has a weak variation in vorticity, or is even irrotational. A solid body generates vorticity as it moves through the fluid (at high Reynolds number), and carries the vorticity along with it.

The important difference with this and the Euler equations, is the addition of the viscosity term $\nu\frac{\partial^2 u}{\partial y^2}$, which is due to a strong gradient of u in the direction normal to the body, and close to it. For boundary layers this term can be very large.

It is worth noting that boundary layer theory cannot be rigorously derived, merely hypothesised.

In practice, the edge of the boundary layer is often to be the position where the flow speed is very close to the outer flow speed, so $y = \delta$ where $u = 0.99U_0$. The general idea is that the flow external to the boundary layer is given by a solution to Eulers equation, and this provides the outer boundary conditions for the layer.



(a) Ideal flow.

(b) Flow with boundary layer.

Figure 7.1: The velocity profiles past a thin plate. (a) shows the ideal flow, where the profile is completely unaffected by the plate; (b) shows a more realistic flow, where the passage of the flow past a plate induces a boundary layer, and the velocity profile is disturbed. In case (b) we have a wake forming because of the disturbance of the flow.

With reference to Figure (7.1), we see the difference in velocity profiles for ideal flow and a flow allowing a boundary layer, past a thin plate. With reference to Figure (7.2), we see the velocity profiles for two types of flow past a sphere. The high R flow separates, forming a boundary layer, thickness δ.

7.1 2D Boundary Layers

Consider a 2D steady flow, as in Figure (7.3). We have length scales δ, L for the boundary layer and length of body, respectively; and velocity scales U, V for the horizontal and vertical directions, respectively.

Let us consider the appropriate scales in the NS equations.

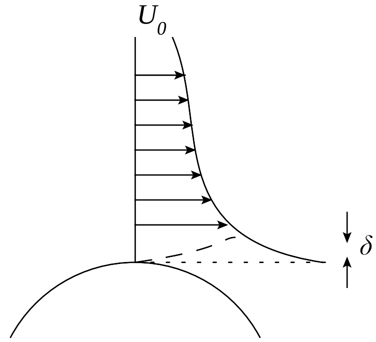


Figure 7.2: The difference in velocity profiles for inviscid (dotted line) and high R (dashed line) flows across a sphere. As the flow gets to the "edge" of the sphere, the flow will either continue in a straight line (i.e. the inviscid flow case) or will separate with some boundary layer δ.

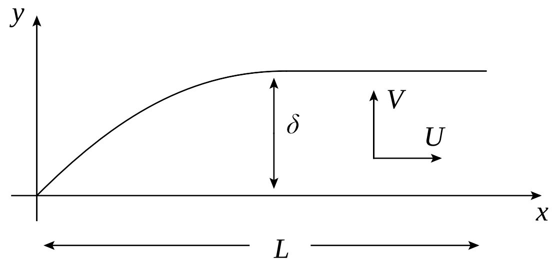


Figure 7.3: A 2D boundary layer, thickness δ, above a surface (at $y = 0$).

Recall (7.2), where

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{R}}.$$

Now, the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

has its terms having the scales

$$\sim \frac{U}{L} \sim \frac{V}{\delta},$$

and therefore we have

$$V \sim \frac{U\delta}{L}. \tag{7.3}$$

This is because the terms of the continuity equation must be of the same order, so that they add and cancel; since fluid entering the boundary layer must exit, even though the boundary layer thickness is not constant. Therefore, (7.3) tells us that the flow speed "up", V, is small compared to the flow speed "across", U, when the boundary layer is thin (i.e. when δ is small). We can use this to scale the NS equations. So, the x-component of the Navier-Stokes equations reads

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

after assuming steady flow. The first term is $\sim U^2/L$, the second term is $\sim VU/\delta$, fourth term is $\sim \nu U/L^2$ and the fifth term $\sim \nu U/\delta^2$. Using (7.3), we see that the order of the second term can be written $\sim VU/\delta \sim U^2/L$, which is the same order as the first term. Therefore, we see that the inertia terms balance. Now, $V < U$, but $\frac{\partial u}{\partial y} > \frac{\partial u}{\partial x}$. Therefore, we neglect the fourth term relative to the fifth term (Notice that this can be seen due to $L \gg \delta$).

Now, outside of the boundary layer, we have our standard flow with

$$U_0\frac{\partial U_0}{\partial x} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x}.$$

Therefore, the NS equations reduce to the boundary layer equations,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_0\frac{dU_0}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{7.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{7.5}$$

Hence, we have two equations in two unknowns u, v. The equations can be solved if the outer-flow speed U_0 is specified, so that the boundary conditions for the layer are found (i.e. solving Eulers equation for the free streaming outer-flow).

The solution to the boundary layer equations tends to be found by introducing a stream-function, and seeking similarity solution (that is, solutions with the same form for all x, but scales with y). This then gives a single non-linear equation, which is solved numerically.

There are three useful classifications for the boundary layer equations; three useful situations.

Negative Pressure Gradient: Accelerating Flow Here we have a negative pressure gradient, and accelerating flow (i.e. the outer flow field has a reduction in pressure as you go "across");

$$\frac{dU_0}{dx} > 0, \quad \frac{dp_0}{dx} < 0$$

This is called a favourable pressure gradient. See Figure (7.4) for an example. The reduction in pressure gradient above the layer means that the layer accelerates, and becomes thinner; the thickness is reduced downstream. Therefore, such a boundary layer flow is stable.

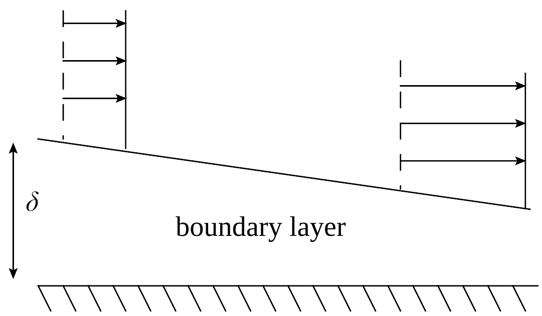


Figure 7.4: A 2D boundary layer, thickness δ; the flow in the boundary layer accelerates as the pressure gradient above the layer drops.

Zero Pressure Gradient This corresponds to a uniform outer-flow field, and hence a uniform boundary layer – for example, flow over a flat plate, the so-called *Blasius flow.* The boundary layer looks like Figure (7.3). The boundary conditions are $u = v = 0$ at $y = 0$ and $u \to U_0$ as $y/\delta \to \infty$. Now, recall that $\delta \sim \sqrt{\nu}$. Hence, $u \to U_0$ as $y/\nu^{1/2} \to \infty$.

Positive Pressure Gradient Here consider

$$\frac{dU_0}{dx} < 0, \quad \frac{dp_0}{dx} > 0;$$

so that we have an adverse pressure gradient, and decelerated outer flow field. What happens here, is that the boundary layer separates from the surface, and produces eddies. The boundary layers thickness downstream thickens, meaning that the boundary layer is unstable. The boundary layer flow then modifies the Euler outer flow.

8 Hydrodynamic Instability & Transition to Turbulence

In previous sections we have derived exact solutions to the Navier-Stokes equations of motion: this led to exactly predictable behaviour of a system. These systems are somewhat unphysical due to the apparent exact knowledge of initial conditions (as well as the unphysical simplifications we used to get the soluble NS equations). However, with most of our solutions, we gave ranges of validity, for the Reynolds number.

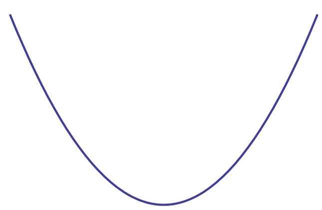
Suppose we have an exact solution to the NS equations, and we add on some infinitesimal perturbation,

$$A(t) = A_{\text{exact}} + \varepsilon e^{\lambda t}, \quad \varepsilon \ll 1. \tag{8.1}$$

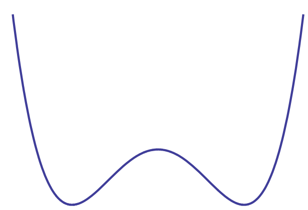
Then, if $\lambda < 0$, the solution will stabilise on $A_{\text{exact}} + \varepsilon$. Conversely, if $\lambda > 0$, the solution will be unstable, and will exponentially diverge from A_{exact}. We can relate λ as a distance from some critical Reynolds number R_{c},

$$\lambda = R - R_{\text{c}}. \tag{8.2}$$

Hence, we can see that for $R < R_{\text{c}}$, the exponent λ in the perturbation term is negative, and the perturbation term decays exponentially. Hence, the solutions are stable. Conversely, if $R > R_{\text{c}}$, the exponent is positive, and the solution is unstable.



(a) The local minimum is stable.



(b) The previous local minimum becomes a maximum. The system must then choose which of the new minima to occupy.

Figure 8.1: Changing the stability; which will correspond to $\lambda > 0$ or $\lambda < 0$.

If the fully nonlinear system of NS equations are considered in the perturbation equation, the stability approach can lead to a sequence of instabilities which lead to more and more complicated flows – as in Figure (8.1). Effectively what this means, is that as the Reynolds number is increased, different types of instability will come into play – the example in the figure depicts how the local minima of the system of equations could change with Reynolds number.

Some approaches only consider linearised equations; in this case, only first order instability is strictly valid.

An example is flow past a cylinder. For $R < 1$, we get linear Stokes flow, where the flow lines are symmetric. As the Reynolds number is increased, $1 < R < 30$, an asymmetric wake forms. The general idea is this: as the Reynolds number is increased, past certain limits, new types of instabilities come into play. These instabilities effectively add new elements of unpredictability. This loss of predictability is not due to a breakdown of the deterministic NS equations; for example, the non-linear oscillator problem is non-linear, and unpredictable, but the equations of motion are exact. The problem comes in the level of accuracy that the initial conditions are known.

The long term unpredictability is chaos: an instability that becomes "amplified".

We gain insight into turbulent flows by studying its origins using known exact solutions. This method does not always produce instabilities: circular Couette flow does, but planar does not. Similarly, plane Poiseuille flow has an instability, but circular does not.

8.1 Linear Stability

We can analyse a simple model, devised by Landau and Stuart and Wilson. Their equation is

$$\dot{A} = \lambda A - A^3, \tag{8.3}$$

where A is the amplitude of the solution, which may be complex, and $\lambda = R - R_c$. So, let us consider steady solutions, with $\dot{A} = 0$, and look at their stability. So, the above equation just becomes

$$A(\lambda - A^2) = 0.$$

This has two solutions: $A = 0$ or $A = \pm\sqrt{\lambda}$. We now restrict ourselves to real amplitudes, $A \in \mathbb{R}$. So, for $A = 0$, we have no restriction on λ. However, if $A = \pm\sqrt{\lambda}$, we must have that $\lambda > 0$.

We can plot these steady solutions on a bifurcation diagram, as in Figure (8.2). With reference to the figure, we denote the solution $A = 0$ as the trivial solution, and we note that new non-trivial stable solutions exist for $\lambda > 0$. We have marked it on, but the solution with $A = 0, \lambda > 0$ is unstable.

In practice, one of the states will be preferred over the other, as in Figure (8.3). This could be the case if there is some imperfection in equipment, meaning that a ball will tend to roll one way rather than the other. This could be modeled by simply adding a constant to (8.3), so that $\dot{A} = \lambda A - A^3 + \delta$.

To test the stability of the trivial state, we add a perturbation to the static solution,

$$A(t) = 0 + \varepsilon(t),$$

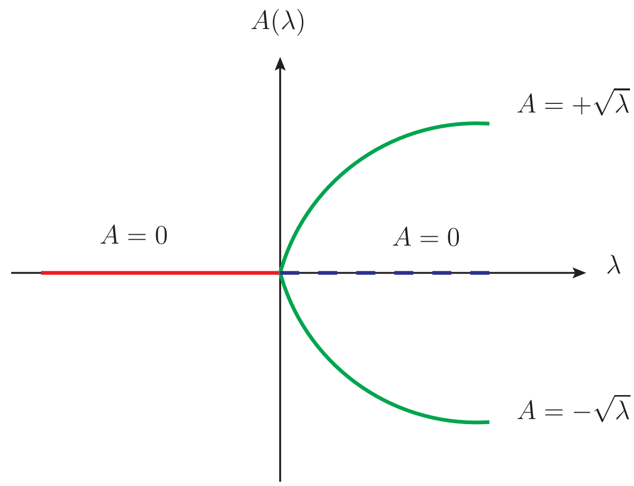


Figure 8.2: Perfect supercritical pitchfork bifurcation. The solution amplitude, as a function of $\lambda = R - R_c$. The solution $A = 0$ for $\lambda > 0$ is unstable – we use the notation that full lines denote stability, and dashed lines instability.

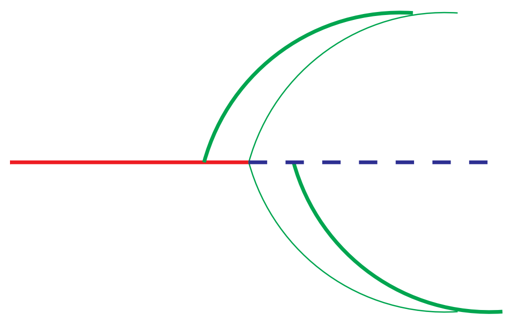


Figure 8.3: An imperfect bifurcation (thick lines), relative to the perfect one (thin lines). This is the case if a particular state is preferred over the other.

where $\varepsilon(t)$ is some small amplitude of a perturbation. We can then substitute this "ansatz" into (8.3), to easily see that

$$\dot{\varepsilon} = \lambda(0 + \varepsilon) - (0 + \varepsilon)^3.$$

To first order in perturbation only, this is just

$$\dot{\varepsilon} = \lambda\varepsilon,$$

where $\dot{\varepsilon} \equiv \frac{\partial \varepsilon}{\partial t}$. The solution to this is just

$$\varepsilon(t) = \varepsilon_0 e^{\lambda t}.$$

Hence, we see that the perturbation will grow exponentially if $\lambda > 0$, and will exponentially decay if $\lambda < 0$. Hence, there is a change in stability of the trivial solution at $R = R_c$.

Let us test the stability of the $A = +\sqrt{\lambda}$ solution. So, substituting into (8.3), we have

$$\begin{aligned}\dot{\varepsilon} &= \lambda\left(\sqrt{\lambda}+\varepsilon\right) - \left(\sqrt{\lambda}+\varepsilon\right)^3 \\ &= \lambda^{3/2} + \lambda\varepsilon - \lambda^{3/2} - 3\lambda\varepsilon + \mathcal{O}(\varepsilon^2) \\ &= -2\lambda\varepsilon.\end{aligned}$$

Hence, the solution to this is just

$$\varepsilon(t) = \varepsilon_0 e^{-2\lambda t}, \quad \lambda > 0.$$

This is an exponentially decaying solution. Therefore, the non-trivial $\sqrt{\lambda}$ solution is linearly stable (as can be similarly shown for the $-\sqrt{\lambda}$ solution).

Therefore, our analysis of (8.3) has shown that there is an exchange of stability between the trivial state and a pair of non-trivial states at $R = R_c$.

Let us consider a second model,

$$\dot{A} = \lambda A + A^3. \tag{8.4}$$

Then, static solutions have amplitudes being the roots of

$$A(\lambda + A^2) = 0.$$

Hence, we easily see that we have the trivial state $A = 0$, and two non-trivial states $A = \pm\sqrt{-\lambda}$. Thus, for real amplitudes, we require $\lambda < 0$. So, let us have solution $A(t) = A_0 + \varepsilon(t)$, where A_0 is one of the three static solutions, and $\varepsilon(t)$ is the amplitude of some time dependent perturbation from the solution. First consider the trivial solution, $A_0 = 0$, so that upon substitution into (8.4), we have that

$$\dot{\varepsilon} = \varepsilon\lambda \quad \Rightarrow \quad \varepsilon(t) = \varepsilon_0 e^{\lambda t}.$$

Now, this describes a solution that is stable for $\lambda < 0$, and unstable for $\lambda > 0$. If we now take $A_0 = +\sqrt{-\lambda} = i\sqrt{\lambda}$, the equation (8.4) becomes (to first order in perturbation only),

$$\dot{\varepsilon} = -2\varepsilon\lambda \quad \Rightarrow \quad \varepsilon(t) = \varepsilon_0 e^{-2\lambda t}.$$

Now, these solutions only existed for $\lambda < 0$, and are hence always unstable.

See Figure (8.4) for the bifurcation diagram corresponding to this system.

Now, the bifurcation diagram for (8.4) shows that there are no stable solutions for $A \neq 0$. We can modify the equation into

$$\dot{A} = \lambda A + A^3 - A^5, \tag{8.5}$$

which, with reference to its diagram, Figure (8.5), we see that there exists stable solutions for $A \neq 0$. In this case, we see that for a given λ there can be 3 stable solutions.

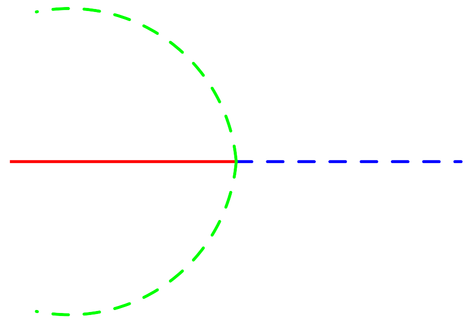


Figure 8.4: Subcritical pitchfork bifurcation describing the stability of solutions to (8.4).

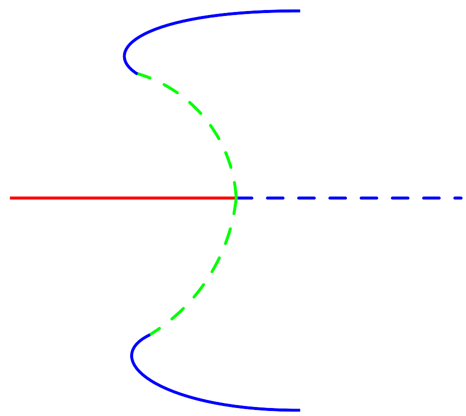


Figure 8.5: Pitchfork bifurcation describing the stability of solutions to (8.5).

8.1.1 Hopf Bifurcation

In contrast to the pitchfork bifurcations, which gave rise to two stable solutions from a single solution, the Hopf bifurcation gives rise to periodic oscillatory solutions from a steady solution. This essentially corresponds to a periodic transition between the two states of the bifurcation.

Such a system of equations that has such a behaviour is

$$\dot{r} = r(\lambda - r^2), \quad \dot{\theta} = \omega. \tag{8.6}$$

With reference to Figure (8.6), we see the Hopf bifurcation diagram; which is a single periodic orbit with frequency ω. Such a bifurcation can be realised by increasing the Reynolds number R of a flow about a cylinder. At low R, the flow will be steady, but as R is increased past some R_c, the flow becomes oscillatory. Plane Poiseulle flow is an example of a subcritical Hopf bifurcation.

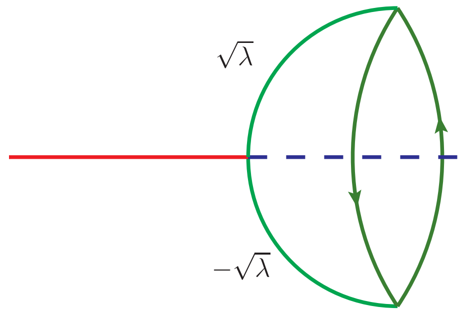


Figure 8.6: The supercritical Hopf bifurcation diagram describing the system (8.6). Notice the periodic oscillation between the two states.

8.2 Landau Theory for Onset of Turbulence

We have seen from our simple linear analysis that in passing some critical Reynolds number, a steady flow can become oscillatory. So we could consider the onset of turbulence – which is the essentially random unpredictable motion of a fluid – as a result of passing many of these types of critical values.

Let us propose that there is a critical Reynolds number R_c, such that at $R = R_c$, periodic instability comes into affect, and grows as

$$|A|_{\max} \propto \sqrt{R - R_c}.$$

That is, the instability becomes greater in amplitude with Reynolds number. The corresponding velocity field is

$$\mathbf{u} = \mathbf{f}(x, y, z)e^{-i(\omega_1 t + \beta_1)},$$

where ω_1 is the first oscillatory frequency, and β_1 is the initial phase. The Landau theory is that as R is increased, there is a further instability with ω_2, β_2, and subsequent instabilities ω_i, β_i; with each new frequency arising from a smaller increment in R, so that at some finite R an infinite number of modes are excited with random phase. This infinite mode excitement is equivalent to turbulence.

See Figure (8.7) for a schematic view of the amplitude of instabilities in the Landau theory. Such a velocity distribution will then have

$$\mathbf{u}(x, y, z) = \sum_n \mathbf{A}_n(x, y, z)e^{-i\sum_{j=1}^{n}(\omega_j t + \beta_j)}. \tag{8.7}$$

Hence, turbulence is an infinite sum over all possible modes, with a random phase. This is consistent with the statistical description of turbulence.

We must ask however, is this viewpoint correct? Do all instances of the onset of turbulence in a fluid occur after an infinite sequence of instabilities (i.e. as in the Landau theory), or

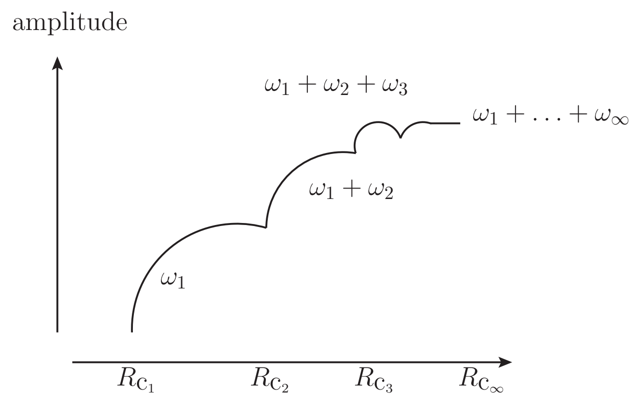


Figure 8.7: Schematic of the Landau theory of the onset of turbulence.

is there just a small number of instabilities (which is the methodology from chaos theory). We can look at Taylor-Couette flow as an example. Consider concentric cylinders, the inner having radius r_1 and rotating with angular velocity Ω_1; the outer has radius r_2 with angular velocity Ω_2. Assuming steady flow and an azimuthal components only, the Navier-Stokes equations reduce to

$$\frac{\rho v^2}{r} = \frac{d\rho}{dr}, \quad 0 = \nu \left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} \right).$$

The first is the statement that pressure is balanced by the centripetal force generated by the rotation. We use the boundary conditions that $v(r = r_1) = \Omega_1 r_1$ and $v(r = r_2) = \Omega_2 r_2$. Then, the first equation is solved to give the pressure distribution, and the second the velocity distribution. One finds

$$v = Ar + \frac{B}{r},$$

where

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad B = \frac{(\Omega_2 - \Omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2}.$$

Rayleigh's Criterion Consider an axially symmetric inviscid rotating flow. Let r be the distance from the axis, p the pressure, ρ the density and v the velocity which is a prescribed function of r. Let the fluid move in circular paths, so that we take toroidal elements (i.e. fluid rings). The motion will result in a balance in centripetal force $\rho v^2/r$ and pressure $-dp/dr$. Consider a ring at radius r_1, with angular velocity v_1. Suppose we displace the ring to $r_2 > r_1$. Then, angular momentum is conserved, so that the new velocity is

$$v_1' = \frac{r_1 v_1}{r_2}.$$

For equilibrium at the new radius, we need a centripetal force $\rho v^2/r$,

$$\frac{\rho r_1^2 v_1^2}{r_2^3},$$

but, a pressure gradient at r_2 supplies an inward force

$$\frac{\rho v_2^2}{r_2}.$$

This, in general, is not equal to the centrifugal force required to balance the newly formed fluid ring. Thus, if

$$\frac{\rho v_2^2}{r_2} > \frac{\rho r_1^2 v_1^2}{r_2^3}, \tag{8.8}$$

the fluid ring will be forced back to where it came from. However, if

$$\frac{\rho v_2^2}{r_2} < \frac{\rho r_1^2 v_1^2}{r_2^3}, \tag{8.9}$$

the fluid ring will continue to move outwards.

Therefore, the flow is stable if (8.8) holds,

$$(r_2 v_2)^2 > (r_1 v_1)^2, \tag{8.10}$$

and unstable if (8.9) holds,

$$(r_2 v_2)^2 < (r_1 v_1)^2. \tag{8.11}$$

Notice that rv is infact the circulation of the flow. Therefore, we can now state Rayleigh's Criterion:

A flow is stable if the square of the circulation increases outwards, and unstable if it decreases outwards.

We can apply this to the rotating Couette flow. Let Ω_1 and Ω_2 denote the angular velocities of the inner and out cylinders, of radii r_1, r_2 respectively. If the cylinders rotate in the same direction, the flow is unstable if $(\Omega_1 r_1)^2 > (\Omega_2 r_2)^2$. For example, if the inner cylinder rotates and the outer cylinder is stationary, the flow is always unstable. If the outer cylinder is stationary, and the gap between the cylinders is small, there is a linear shear problem. The action of viscosity is to stabilise the flow until the critical speed is reached.

Suppose we now make the cylinders rotate in opposite directions. Circulation now only decreases in part of the flow – this modifies the stability criterion into

$$\frac{d}{dr}|\Omega r^2| < 0 \quad \Longleftrightarrow \quad \text{instability.}$$

This can be seen in a system with translational invariance (such as infinite cylinders), where there will be multiple bifurcations; initially cells form, then Hopf bifurcations produce waves within the cells.

8.3 Turbulence

Chaos is a situation where a finite number of bifurcations are required for disorder – this gives temporal disorder, and is not turbulence. The Landau theory stated that a infinite sequence of bifurcations leads to turbulence. Now, the idea of a sequence of instabilities does not always work – for rotary Taylor-Couette flow it does work, but for pipe flow or plane Couette flow, the idea does not work; both being linearly stable, but able to form disordered flow.

The problem of turbulence is often called the last great unsolved problem of classical physics, and is of fundamental importance as most flows are turbulent – as a simple example, flow from a tap. The nature of the problem forces a descriptive description. If the flow rate is small, the laminar flow is smooth, and is a derivable exact solution of the NS equations. However, if the flow rate is increased, there is a sudden transition to turbulent (rough) flow – so rough, an empirical description is required. We could ask however, if the Reynolds number is increased far enough, does the flow become completely random – the answer, fairly obviously, is no.

The probability distribution of the is non-Gaussian, but the flow is not completely random, as it contains structures (such as eddies). Thus, statistical properties are useful, such as the mean velocity distribution.

Ideas can be explored using the special case of grid generated turbulence, whereby a regular grid is moved through a fluid. Suppose we have a grid with spacing m. Then, initial eddies of size m cascade down to smaller eddies as they flow. Consider in Fourier space – the energy is initially at large scales, but changes to smaller scales – an inertial range of scales.

8.3.1 Energy Cascade

The general idea for how big eddies supply energy to smaller eddies is this. Energy is transfered down from large to small eddies, until the action of viscosity on the smallest scales – k_{d}. Large eddies interact with mean flow; for example, in a pipe of radius r, driven by mean flow, dissipating eddies are of order $0.01r$. The transfer of energy takes place via vortex stretching, which is a 3D process. Viscosity is relatively unimportant in turbulent flows. The physical mechanisms of energy transfer in turbulence is:

- Cascade of instabilities: energy large to smaller scales, giving rise to greater complexity.
- Small scales extract energy from mean flow.
- Vortex lines stretch, increasing vorticity, and tangle; hence, a loss of predictability.

We can consider the flow field as $\mathbf{u} + \mathbf{u}_f$, where the first term is a mean velocity and the second the fluctuations. This gives rise to an extra self-interaction term in the NS equation,

$$\mathbf{u}_f \frac{\partial \mathbf{u}_f}{\partial \mathbf{x}}.$$

8.4 The Kolmogorov Spectrum

An hypothesis for dealing with the energy cascade comes from a fairly simple dimensional analysis.

Consider a energy spectrum $E(k)$ as a function of wavenumber k in the inertial range (i.e. on small scales). Then, $E(k)dk$ is the average turbulent energy per unit mass between $k \to k + dk$. To get a feel for $E(k)$, suppose we define the kinetic energy per unit volume,

$$\Phi \equiv \frac{1}{2}\rho\bar{q}^2,$$

where the average velocity is

$$\bar{q}^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2.$$

Hence,

$$\frac{\Phi}{\rho} = \frac{1}{2}\bar{q}^2 = \int_0^\infty E(k)\,dk.$$

The dimensions of $E(k)$ can be computed quite simply,

$$[E] = \frac{[\bar{q}^2]}{[k]} = \frac{L^2T^{-2}}{L^{-1}} = L^3T^{-2}.$$

Now, suppose we define the energy dissipation $\mathcal{E}$, which is the energy flow per unit mass per unit time (i.e. down the scales in inertial range, until reach k_d^{-1}, the size of fluid elements themselves). Thus, by dimensions, we have

$$[\mathcal{E}] = ML^2T^{-2}T^{-1}M^{-1} = L^2T^{-3}.$$

We now assume that the dependence of $E(k)$ upon $\mathcal{E}$ and k is

$$E(\mathcal{E}, k) = C\mathcal{E}^\alpha k^\beta,$$

where we can deduce α and β via dimensional arguments,

$$[E] = [\mathcal{E}]^\alpha[k]^\beta.$$

Hence,

$$L^3T^{-2} = (L^2T^{-3})^\alpha(L^{-1})^\beta,$$

so by equating the exponents of L and T, we can easily see that

$$\alpha = \frac{2}{3}, \quad \beta = -\frac{5}{3}.$$

Therefore,

$$E(\mathcal{E}, k) = C\mathcal{E}^{2/3}k^{-5/3}, \tag{8.12}$$

where C is a dimensionless constant. This has been confirmed by experiment, and atmospheric flow experiments have found that $C = 1.5$. The agreement is not perfect at high k (i.e. very small scales), where there is no real consensus as to a reason.

There is a subtlety with this. Dissipation is not isotropic at small scales, as we assumed above.

8.5 Reynolds Stresses

Our aim is to show that small fluctuations lead to large flow resistance in turbulent flows via Reynolds stresses.

Consider an infinitesimal block of fluid, $dxdydz$. Then, the rate at which the x-component of momentum passes through $dydz$ is the mass flux times the velocity,

$$= \rho U dydz U.$$

Now, in turbulent flow, we represent the motion as an averge plus some fluctuation which does not have an average,

$$U = u + u_f,$$

where $\bar{u}_f = 0$. Hence, the above expression for momentum rate becomes

$$\begin{aligned} \rho U dydz U &= \rho U^2 dydz \\ &= \rho(u + u_f)^2 dydz \\ &= \rho(u^2 + 2uu_f + u_f^2)dydz, \end{aligned}$$

but taking the average of this quantity removes the middle term from the brackets,

$$\rho(u^2 + \bar{u}_f^2)dydz.$$

Therefore, we see that the rate of momentum has an additional term due to the fluctuations with zero mean. The point of this extra term is that the non-linear term in the NS equations gets modified, to give extra stresses the so-called Reynolds stresses: $\sim \rho\bar{U}^2$.

In pipe flow, experiment tells us that turbulent fluctuations are $\approx \pm 10\%$ about the mean. Hence, the Reynolds stresses are $\approx 0.001\rho U^2$. Velocity gradients in a pipe are $\sim U/r$, so that viscous stresses are $\sim \mu U/r$. Hence, the ratio is

$$\frac{\text{Reynolds stresses}}{\text{viscous stresses}} = 0.001\frac{Ur}{\nu} = 0.001R.$$

Thus, for a flow with $R = 10^6$, the Reynolds stresses are about 1000 times the viscous stresses.