The Early Universe:
Quick Guide

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Abstract

This is a quick guide – a summary – of The Early Universe course at the University of Manchester, taught by R.Battye between Jan '09 and May '09. These summary notes are based upon his lecture notes. A copy of the full lecture notes, on this topic, may be found at `www.jpoffline.com`.

Keywords:

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I. THE FRW UNIVERSE

The FRW **astronomers metric** is

$$
\begin{aligned}
ds^2 &= dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \\
&= a^2(t)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],
\end{aligned}
$$

where we have introduced the **conformal time**

$$a(t) = \frac{dt}{d\eta}.$$

The **curvature constant** k can be interpreted in terms of the spatial parts of this metric,

$$k = \begin{cases} 1 & \text{closed,} \\ 0 & \text{flat,} \\ -1 & \text{open.} \end{cases}$$

Einstein's equation is just

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + g_{\mu\nu}\Lambda,$$

where we have included the **cosmological constant** Λ. After inserting the FRW metric into Einstein's equation, one is able to derive the equations of motion; by computing the Christofell symbols from the FRW metric, and inserting into the continuity equation, one is able to derive the **fluid equation**,

$$
\begin{aligned}
\textbf{Friedmann}: &\quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \\
\textbf{Raychaudhuri}: &\quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right), \\
\textbf{Fluid equation}: &\quad \dot{\rho} = -3H\left(\rho + P\right).
\end{aligned}
$$

We define the **critical density** to be the density that makes the universe flat,

$$\rho_{\mathrm{c}} = \frac{3H^2}{8\pi G}.$$

Redshift and **temperature** are related to the **scale factor** via

$$1 + z = \frac{1}{a}, \quad T = \frac{T_0}{a}.$$

A radial null geodesic reduces to just

$$dr = \frac{dt}{a(t)}.$$

The **luminosity distance** is defined

$$d_{\rm L} = (1+z)r(z),$$

so that the **observed flux** relates to the emitted luminosity via

$$F_{\rm obs} = \frac{L_{\rm emit}}{4\pi d_{\rm L}^2}.$$

Similarly, we define the **angular diameter distance**,

$$d_{\rm A} = \frac{r(z)}{1+z},$$

so that the **physical comoving size** becomes

$$R = d_{\rm A}\theta.$$

A. Observations

The current **temperature of the cosmic microwave background** is

$$T_{\rm CMB,0} = 2.728 \pm 0.004{\rm K},$$

and is the best black-body. The **Hubble parameter** is written

$$H_0 = 100h{\rm km\ sec}^{-1}{\rm Mpc}^{-1},$$

where current observations suggest

$$h = 0.72 \pm 0.08.$$

The value h is measured using **standard candles** – Cephid variables with known intrinsic luminosity. The **deceleration paramter** is

$$q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = -0.55,$$

which means that the universe is currently accelerating.

B. Anisotropies & Density Contrast

The **anisotropies** of the CMB (i.e. the deviations from exactly black-body) are decomposed in terms of spherical harmonics,

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{\ell=2}\sum_{m} a_{\ell m} Y_{\ell m}(\theta,\phi),$$

where we have ignored the monopole and dipole moments due to the **uniform background** and **earths motion**, respectively. The spherical harmonics are orthogonal,

$$\int d\Omega\, Y_{\ell m} Y^{*}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'},$$

which allows us to compute the complex coefficients

$$a_{\ell m} = \int d\Omega\, Y^{*}_{\ell m}(\theta,\phi)\frac{\Delta T}{T}(\theta,\phi).$$

The **angular correlation function** gives the expected value of finding two objects of the same angular size, in different directions;

$$\begin{aligned} C(\theta) &= \left\langle \frac{\Delta T}{T}(\hat{n})\frac{\Delta T}{T}(\hat{n}') \right\rangle \\ &= \frac{1}{4\pi}\sum_{\ell}(2\ell+1)C_\ell P_\ell(\cos\theta), \end{aligned}$$

where

$$C_\ell = \frac{1}{2\ell+1}\sum_{m}|a_{\ell m}|^2.$$

In a typical plot of the anisotropies of the CMB, we generally plot

$$\ell \quad \text{against} \quad \frac{\ell(\ell+1)C_\ell}{2\pi}T^2_{\text{CMB}}.$$

The **density contrast** can be computed, and written as a Fourier transform,

$$\begin{aligned} \delta(\mathbf{x},t) &= \frac{\rho(\mathbf{x},t)-\bar{\rho}(t)}{\bar{\rho}(t)} \\ &= \frac{V}{(2\pi)^3}\sum_{\mathbf{k}}\delta_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}. \end{aligned}$$

Hence, inverting the transform, one has the amplitude of a mode,

$$\delta_{\mathbf{k}} = \frac{1}{V}\int d^3x\, \delta(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The **power spectrum** is just the modulus squared of this amplitude,

$$P(k,t) = |\delta_k(t)|^2 .$$

The **two point correlation function** is defined as

$$\xi(\mathbf{r}) = \frac{1}{V}\int d^3x\, \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}),$$

and one can show that this is just the Fourier transform of the power spectrum,

$$\xi(r) = \frac{V}{(2\pi)^3}\sum_{\mathbf{k}} P(k)e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

C. Decoupling

From black-body theory, one can compute that for a radiation fluid in thermal equilibrium,

$$\rho_{\mathrm{r}} = \frac{\pi^2}{30}gT^4 = \frac{3H^2}{8\pi G},$$

where the second equality follows from the Friedmann equation. This is rearranged into

$$H = \sqrt{\frac{4\pi^3}{45}}g^{1/2}\frac{T^2}{m_{\mathrm{pl}}},$$

where the **Planck mass** is

$$m_{\mathrm{pl}}^2 = \frac{1}{G}.$$

The degeneracy g in the Hubble rate H is given by

$$g = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^4,$$

for species in thermal equilibrium. Similarly, the entropy is

$$S = \frac{2\pi^2}{45}g_{\mathrm{s}}(aT)^3,$$

where

$$g_{\mathrm{s}_i} = \frac{g_i}{T_i}.$$

Hence, either side of a particle dropping out of thermal equilibrium, we have

$$g_{\mathrm{s}}T^3\big|_{\text{before}} = g_{\mathrm{s}}T^3\big|_{\text{after}} .$$

Decoupling occurs when the interaction rate drops below the Hubble rate,

$$\text{decoupling}: \quad \Gamma_{\text{int}} = H,$$

where, for neutrinos, we have

$$\Gamma_{\text{int}} = G_{\text{F}}^2 T^5.$$

Hence,

$$\frac{\Gamma_{\text{int}}}{H} = \left(\frac{T}{1\text{MeV}}\right)^3,$$

so that neutrino decoupling happens at a temperature $T_{\text{dec}} \approx 1\text{MeV}$. This allows the computation of

$$\Omega_{\text{r}} h^2 = 4.2 \times 10^{-5}.$$

D. Species Summary

Here we provide a table of the parameter dependencies for the main species in the universe. The equation of state is such that $P = w\rho$.

	$\rho(a)$	$a(t)$	w
matter	a^{-3}	$t^{2/3}$	0
radiation	a^{-4}	$t^{1/2}$	1/3
Λ	const	e^{Ht}	-1

II. STRUCTURE FORMATION

Structure formation is interested in the processing of an initial spectrum, by the evolution of an expanding universe.

A **transfer function** describes the processing of an initial power spectrum;

$$\delta_{\mathbf{k}}(t_0) = T(k)\delta_{\mathbf{k}}(t_i), \quad P_0(k) = T^2(k)P_i(k).$$

The **Newtonian fluid equation**, in an expanding universe, describes the evolution of the amplitude of perturbation modes δ, and reads

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\left(4\pi G\rho - \frac{c_{\text{s}}^2 k^2}{a^2}\right),$$

where the **sound speed** is

$$c_{\mathrm{s}}^2 = \frac{\partial P}{\partial \rho}.$$

We must note that for temperatures lower than its rest mass, CDM is collision-less, and so $c_{\mathrm{s}}^2 = 0$. For temperatures above decoupling temperatures, the fluid can be treated as just a photon fluid, in which case $c_{\mathrm{s}}^2 = 1/3$. The fluid equation can be solved in a few cases.

- *No expansion*; $a = 1, \dot{a} = 1$. In this case, we compute an oscillatory solution for the perturbation. However, we define a **Jeans length** to distinguish real and complex solutions,

  $$\lambda_{\mathrm{J}} = c_{\mathrm{s}}\sqrt{\frac{\pi}{\rho G}}.$$

  For scales $\lambda > \lambda_{\mathrm{J}}$, the perturbations δ grow **exponentially**. For scales $\lambda < \lambda_{\mathrm{J}}$, the perturbations oscillate, supported by pressure. Notice that if just CDM, then $c_{\mathrm{s}} = 0$, and hence perturbations grow on all scales.

- *CDM domination*; with $c_{\mathrm{s}}^2 = 0, a \propto t^{2/3}$. Upon substitution into the fluid equation, we then make a power law ansatz $\delta \propto t^p$, from which we can easily find that

  $$\delta \propto t^{-1}, \quad \delta \propto t^{2/3}.$$

  The first is a decaying mode, which we neglect; the second is a growing mode, which is infact $\delta \propto a$.

- Λ *domination*; with $c_{\mathrm{s}}^2 = 0, a \propto e^t$. In this case, we have that ρ tends to zero very quickly, and we have no growth of δ.

- *Mezaros effect*; taking a mixture of matter and radiation, with $c_{\mathrm{s}}^2 = 0$. This corresponds to evolution through the radiation-matter transition. We define

  $$y = \frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{r}}} = \frac{a}{a_{\mathrm{eq}}},$$

  and change variables so that the fluid equation becomes

  $$\frac{\partial^2 \delta_{\mathrm{m}}}{\partial y^2} + \frac{2+3y}{2y(1+y)}\frac{\partial \delta_{\mathrm{m}}}{\partial y} - \frac{3}{2y(1+y)}\delta_{\mathrm{m}} = 0.$$

  We then solve via a power-law ansatz, easily finding that

  $$\delta_{\mathrm{m}} \propto \frac{2}{3} + y.$$

Now, in the matter era, $a > a_{\rm eq}$ so that $y > 1$. Similarly, in the radiation era, $a < a_{\rm eq}$ so that $y < 1$. Hence,

$$\delta_{\rm m} \propto \begin{cases} \text{const} & \text{radiation era,} \\ a & \text{matter era.} \end{cases}$$

Hence, no growth in the radiation era, only in the matter era. This is the **Mezaros effect**.

A. Cosmology in the Synchronous Gauge

We perturb a conformally Minkowski background metric,

$$ds^2 = a^2 \left(\eta_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu,$$

where the perturbation satisfies the conditions of the **synchronous gauge**,

$$|h_{\mu\nu}| \ll 1, \quad h_{00} = h_{0i} = 0.$$

We then compute the time and space parts of the continuity equation (after computing the new Christofell symbols and energy-momentum tensor),

$$\begin{aligned} \text{time :} \qquad & \delta' = (1+w)\left(\frac{1}{2}h' - \theta\right), \\ \text{space :} \qquad & \theta' + \frac{a'}{a}(1-3w)\theta - \frac{w}{1+w}k^2\delta = 0. \end{aligned}$$

We have used an equation of state $P = w\rho$. δ denotes the spatial gradients of perturbation, and θ the divergence; we have ignored vorticity. Solving these in various cases;

- *CDM domination*; with $w = 0$, the time and space components of the fluid equation just become

  $$\delta'_{\rm m} = \frac{1}{2}h' - \theta_{\rm m}, \quad \theta'_{\rm m} + \frac{a'}{a}\theta_{\rm m} = 0.$$

  We say that the CDM particles are comoving observers, so that we can set $\theta_{\rm m} = 0$. Hence, the time equation becomes just

  $$\delta'_{\rm m} = \frac{1}{2}h'.$$

- *Radiation domination*; with $w = 1/3$, the time and space components become

$$\delta_{\rm r}' = \frac{4}{3}\left(\frac{1}{2}h' - \theta_{\rm r}\right), \quad \theta_{\rm r}' = \frac{1}{4}k^2\delta_{\rm r},$$

from which we can easily derive

$$\delta_{\rm r}'' + \frac{1}{3}k^2\delta_{\rm r} = \frac{2}{3}h''.$$

If we compute the perturbed Einstein equations, we get (among other things),

$$\delta_{\rm m}'' + \frac{a'}{a}\delta_{\rm m} = \frac{3\omega_{\rm r}\delta_{\rm r}}{a^2} + \frac{3\omega_{\rm m}\delta_{\rm m}}{2a}, \quad \omega_i = H_0^2\Omega_i.$$

In the matter era, using $a \approx \frac{1}{4}\omega_{\rm m}\eta^2$, and that $\omega_{\rm r} \ll 1$, we solve with a power law ansatz to find

$$\delta_{\rm m}(k,\eta) = A(k)\left(\frac{\eta}{\eta_i}\right)^2 + B(k)\left(\frac{\eta}{\eta_i}\right)^{-3}.$$

The first term is a growing mode; the second a decaying mode.

Taking all the perturbed Einstein and fluid equations together, we solve using a power series solution, to find that there are two types of initial condition;

$$\delta_{\rm r} - \frac{4}{3}\delta_{\rm m} = \begin{cases} 0 & \text{adiabatic (curvature with } \delta g_{\mu\nu} \neq 0), \\ -\frac{4}{3} & \text{isocurvature (with } \delta g_{\mu\nu} = 0). \end{cases}$$

The **adiabatic** initial conditions occur as a result of **slow-roll inflation**.

	Super-horizon	**Sub-horizon**
Radiation era	$\delta_{\rm m} \propto \eta^2$	$\delta_{\rm m} \propto 1$
Matter era	$\delta_{\rm m} \propto \eta^2$	$\delta_{\rm m} \propto \eta^2$

TABLE I: The dependancies of the matter fluctuation on conformal time, for various cases.

Super-horizon refers to $k\eta \ll 1$, and **sub-horizon** to $k\eta \gg 1$.

We can write the **transfer function** for modes that cross the horizon during the radiation era,

$$T(k) = \left(\frac{\eta_{\rm H}}{\eta_i}\right)^2 .1. \left(\frac{\eta_\Lambda}{\eta_{\rm eq}}\right)^2 .1,$$

or, if we define

$$\frac{k}{k_{\rm eq}} = \frac{\eta_{\rm eq}}{\eta_{\rm H}},$$

the transfer function becomes

$$\text{radiation era}: \quad T(k) = \left(\frac{\eta_\Lambda}{\eta_i}\right)^2 \left(\frac{k}{k_{\text{eq}}}\right)^{-2}.$$

Similarly for modes that cross during the matter era,

$$\text{matter era}: \quad T(k) = \left(\frac{\eta_\Lambda}{\eta_i}\right)^2.$$

This shows that **modes that cross in the radiation era are suppressed**. So, if we have an initial spectrum

$$P_i(k) \propto k^{n_{\text{s}}},$$

then this will be processed to give

$$P(k) = T^2(k)P_i(k) = \begin{cases} k^{n_{\text{s}}} & k < k_{\text{eq}}, \\ k^{n_{\text{s}}-4} & k > k_{\text{eq}}. \end{cases}$$

On small scales, **HDM supresses power**, as it becomes collisional; this means that galaxies cannot form in a universe of only HDM. The scale involved is known as the **Silk damping scale**.

The **conformal Newtonian gauge** is another choice of coordinate, such that the metric has the form

$$ds^2 = a^2\left((1+2\psi)dt^2 - (1-2\phi)d\mathbf{x}^2\right);$$

hence

$$h_{00} = 2\psi, \quad h_{ij} = -2\phi\delta_{ij}.$$

We can use this to convert between the two gauges,

$$\psi = \alpha' + \frac{a'}{a}\alpha, \quad \phi = H - \frac{a'}{a}\alpha,$$

where

$$\alpha = \frac{1}{2k^2}\left(h' + 6H'\right).$$

In this gauge, we find that perturbations do not grow on super-horizon scales; thus, growth is observer dependent.

III. FEATURES OF THE CMB

The main features of the cosmic microwave background are:

- **ISW effect** at very low ℓ – due to recent Λ-domination;
- **SW plateau** at slightly larger ℓ – due to varying gravitational potential along the line of sight;
- **Acoustic peaks and troughs** – due to perturbations in the radiation fluid;
- **Damping envelope** – due to photon diffusion, and the coupling of baryons to photons.

The CMB was released once both **recombination** and **decoupling** occured;

- **Recombination**: cool enough for atomic hydrogen to form;

$$p + e^- \leftrightarrow H + \gamma, \quad T_{\rm rec} \approx 0.3\text{eV}.$$

- **Decoupling**: electron-photon Thomson scattering stops;

$$e^- + \gamma \leftrightarrow e^- + \gamma, \quad T_{\rm dec} \approx 0.25\text{eV}.$$

This happens when the Thomson rate becomes the same as the Hubble rate,

$$\Gamma_{\rm T} = n_{\rm e}\sigma_{\rm T} = H.$$

For $T \gg T_{\rm dec}$, photons and baryons are tightly coupled, so that the mean free path of photons is zero. For $T \ll T_{\rm dec}$, mean free path effectively infinite.

As the density is not spatially constant, H is not spatially constant, as $H \propto \rho^{1/2}$. Hence, decoupling happens at different times in different places. These are **density induced fluctuations**. From Bose theory,

$$\rho_{\rm r} \propto T^4 \quad \Rightarrow \quad \delta_{\rm r} = \frac{\delta\rho_{\rm r}}{\rho_{\rm r}} = 4\frac{\Delta T}{T}.$$

In this regime, photons and baryons are tightly coupled, so that we just have $P = \frac{1}{3}\rho$. On small scales this gives an oscillatory solution; on large scales a power-law solution. We solve the fluid equations to find

$$\theta_{\rm r} \sim \sin k, \quad \delta_{\rm r} \sim \cos k.$$

The radiation fluid moves relative to the Hubble flow – hence, **Doppler induced fluctuations**;

$$\frac{\Delta T}{T} = \delta v = \frac{\theta_{\mathrm{r}}}{k}.$$

These effects, taken together, are responsible for the peaks and spacing of the peaks, in the CMB temperature spectrum.

As move out of tight coupling regime, the mean free path of photons increases; this gives the **photon diffusion damping envelope**, as we modify the fluid equation to include a damping term, and so its solution changes;

$$\delta_{\mathrm{r}} \sim e^{-k^2/k_{\mathrm{D}}^2} \delta_{\mathrm{r}}\Big|_{\mathrm{tight}}.$$

The **Sachs-Wolfe** effect comes from considering the varying gravitational potential along the line of sight;

$$\frac{\delta T_{\mathrm{obs}}}{T_{\mathrm{obs}}} = \frac{\delta T_{\mathrm{emit}}}{T_{\mathrm{emit}}} + \int_0^{\eta_0 - \eta_{\mathrm{dec}}} dy \dot{\phi} - [\psi]_{\eta_{\mathrm{dec}}}^{\eta_0}.$$

The first modification is the **integrated SW effect**, and the second the **ordinary SW effect**.

The **angular diameter distance degeneracy** can be used to measure Ω_{k}, by noting the size of the peaks which change if Ω_{k} changes.

The initial spectrum $A_{\mathrm{s}} k^{n_{\mathrm{s}}}$ can be measured – peak height gives the amplitude, the spectral index via red/blue tilting:

- $n_{\mathrm{s}} > 1$ gives the peaks being higher, and blue-tilted;
- $n_{\mathrm{s}} < 1$ gives the peaks being lower, and red-tilted.

IV. INFLATION

A. The Horizon & Flatness Problems/Solutions

From $\Omega = \rho/\rho_c$, with the equation of state $P = w\rho$, one can derive

$$\dot{\Omega} = (1+3w)H\Omega(\Omega - 1).$$

Thus, if $1 + 3w > 0$, the critical point $\Omega = 1$ is **unstable**. One can further derive that

$$\Omega_{\rm eq} - 1 = 10^{-4}(\Omega_0 - 1),$$

so that the universe must have been much flatter in the past than it is now, in order that we observe the present flatness. Thus, the universe suffers from a **fine-tuning** problem.

The **metric distance to horizon** (i.e. the horizon size) is

$$d_{\rm H} = \frac{t}{1-p}.$$

Hence,

$$\frac{d_{\rm H}(t_0)}{d_{\rm H}(t_{\rm rec})} = 6 \times 10^4 \Omega_{\rm m}^{1/2},$$

which means that there should be many disconnected regions in our observable universe; however, the CMB homogeneity implies that the universe has always been in causal contact.

Hence, to summarise:

$$\text{horizon problem} \iff \text{isotropy of the CMB},$$

$$\text{flatness problem} \iff \text{fine-tuning of } \Omega.$$

We therefore invoke **inflation; a period of superluminal expansion, before radiation domination**. Hence, we have that $\ddot{a} > 0$. Acceleration requires $w \leq -1/3$.

Inflation solves these problems by introducing a species with $1 + 3w < 0$, so that $\Omega = 1$ becomes an attractor solution. Hence,

$$\frac{\Omega_0 - 1}{\Omega_0} = 10^{54} e^{-2\mathcal{N}} \frac{\Omega_{\rm start} - 1}{\Omega_{\rm start}},$$

which means that given a high enough $\mathcal{N}$, the universe pulls itself flat enough during inflation. The number of e-foldings is

$$\mathcal{N} = \ln\left(\frac{a_{\rm end}}{a_{\rm start}}\right),$$

describing the number of times the universe has expanded by a factor of e. We need $\mathcal{N} \approx 60$ to solve these problems.

B. Potential Formulation & Slow-Roll

We introduce a Lagrangian for a real scalar field $\phi = \phi(t, \mathbf{x})$, called the **inflaton**,

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),$$

which has energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}.$$

Hence, assuming spatial homogeneity, $\partial_i \phi = 0$, one finds

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\ P_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi). \end{aligned}$$

The **slow-roll conditions** are

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \quad \ddot{\phi} \ll \frac{dV}{d\phi}.$$

Hence, under the slow-roll conditions,

$$\rho_\phi = -P_\phi \quad \Rightarrow \quad w_\phi = -1,$$

which is the required equation of state for inflation.

Computing the Euler-Lagrange equation gives the equation of motion of the inflaton,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0.$$

Using ρ_ϕ as the energy density species, the Friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$$

Under the slow-roll conditions, these two equations become the **slow-roll equations**,

$$3H\dot{\phi} = -\frac{dV}{d\phi}, \quad H^2 = \frac{8\pi G}{3} V.$$

These can be used to derive

$$\dot{\phi} = -\frac{V'}{\sqrt{24\pi G V}}, \quad \ddot{\phi} = \frac{V'}{24\pi G}\left[\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right], \qquad V' = \frac{dV}{d\phi}.$$

The first equation can be used to find $\phi(t)$ for a given potential $V(\phi)$. We then define the **slow-roll parameters**:

$$\epsilon = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \quad \eta = \frac{m_{\rm pl}^2}{8\pi}\frac{V''}{V} - \epsilon.$$

Hence, these can be used to show

$$\frac{\ddot{\phi}}{V'} = \frac{1}{3}\eta, \quad \frac{\dot{\phi}^2}{V} = \frac{2}{3}\epsilon.$$

Therefore, slow-roll requires

$$|\eta| \ll 1, \quad \epsilon \ll 1 \quad \Longleftarrow \quad \text{slow-roll}.$$

We assume that

$$V\left(\phi_{\rm start}\right) = m_{\rm pl}^4.$$

Using all these equations, one can derive

$$\begin{aligned} a(\phi) &= a_{\rm start} e^{8\pi G \int_{\phi}^{\phi_{\rm start}} \frac{V}{V'} d\phi}, \\ \mathcal{N} &= \frac{2\sqrt{\pi}}{m_{\rm pl}} \int_{\phi_{\rm end}}^{\phi} \frac{d\phi}{\sqrt{\epsilon}} \\ &= -\int_{\phi_{\rm end}}^{\phi} \frac{H}{\dot{\phi}} d\phi, \end{aligned}$$

where the last equation is useful for computing $\mathcal{N}_{\rm tot}$, taking $\phi = \phi_{\rm start}$.

C. Reheating

This is a mechanism to restore a relativistic species to domination at the end of inflation. Slow-roll is violated when

$$\epsilon = \frac{3}{2}, \quad \eta = 3, \quad \dot{\phi}^2 = V, \quad \ddot{\phi} = V',$$

in which case one can compute that $\rho + 3P > 0$, which means no more acceleration. Hence, we define the **end of inflation** to be when

$$\epsilon = 1, \quad |\eta| = 1,$$

whichever occurs first.

If we assume that the Standard Model couples to the inflaton, we assume that at the end of inflation all its energy is transferred to relativistic species; where we assume the species is a black-body,

$$\rho_{\mathrm{R}} = V\left(\phi_{\mathrm{end}}\right) = \frac{\pi^2}{30} g T_{\mathrm{R}}^4,$$

where g is the **number of degrees of freedom in the standard model**. Hence, this is rearranged to give the **reheat temperature**

$$T_{\mathrm{R}} = \left(\frac{30}{g\pi^2}\right)^{1/4} \left[V\left(\phi_{\mathrm{end}}\right)\right]^{1/4}$$

Using the **Klein-Gordon** potential $V = \frac{1}{2}m^2\phi^2$ as an example, one can compute

$$\phi_{\mathrm{end}} = \frac{m_{\mathrm{pl}}}{\sqrt{4\pi}}, \quad \phi_{\mathrm{start}} = \frac{m_{\mathrm{pl}}^2}{m}\sqrt{2}.$$

We can use

$$\mathcal{N}_{\mathrm{tot}} = \frac{2\pi}{m_{\mathrm{pl}}^2}\left(\phi_{\mathrm{start}}^2 - \phi_{\mathrm{end}}^2\right)$$

to show

$$m < 0.45 m_{\mathrm{pl}}, \quad T_{\mathrm{R}} = 0.05 m_{\mathrm{pl}}.$$

D. Fluctuations

The quantum mechanical lore is used to imply that the field does not have a well defined position on the potential. We define the scalar **curvature perturbation**

$$\mathcal{R} = H\delta t = \frac{H}{\dot{\phi}}\delta\phi,$$

and from Parseval's theorem,

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{(2\pi)^2} k^3 |\mathcal{R}(k)|^2 \propto \frac{|\delta(k)|^2}{k}.$$

Hence, we make the power-law ansatz

$$\mathcal{P}_{\mathcal{R}}(k) \propto k^{n-1},$$

where n is the **spectral index**. We can then compute that

$$\begin{aligned} \mathcal{P}_{\mathcal{R}} &= \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \\ &= \frac{2^7\pi}{3}\frac{1}{m_{\mathrm{pl}}^6}\frac{V}{V'^2}. \end{aligned}$$

Using the measured $\mathcal{P}_\mathcal{R} = 2.5 \times 10^{-9}$ with the KG potential, to find

$$m \approx 10^{13}\text{GeV}.$$

We have

$$\begin{aligned} n - 1 &= \frac{d\ln \mathcal{P}_\mathcal{R}}{d\ln k} \\ &= \frac{\dot{\phi}}{H\mathcal{P}_\mathcal{R}}\frac{d\mathcal{P}_\mathcal{R}}{d\phi} \\ &= 2\eta - 4\epsilon. \end{aligned}$$

Performing similar calculations, we have a expression for the **tensor perturbations**,

$$\mathcal{P}_G = \frac{64\pi}{m_{\text{pl}}^2}\left(\frac{H}{2\pi}\right)^2 \propto k^{n_{\text{T}}},$$

where the spectral index for such perturbations is

$$n_{\text{T}} = \frac{d\ln \mathcal{P}_G}{d\ln k}.$$

We then define the **tensor-to-scalar ratio**

$$r = \frac{\mathcal{P}_\mathcal{R}}{\mathcal{P}_G} = 16\epsilon.$$

E. Models

We parameterise models into:

- Large field: where $\phi_{\text{start}} \approx m_{\text{pl}}$, and the potential is generally

$$V(\phi) = V_0\left(\frac{\phi}{\mu}\right)^p.$$

- Small field: where the field starts at low values, with the potential being given by

$$V(\phi) = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^p\right].$$

- Hybrid: inflation stops because of coupling to some other field,

$$V(\phi) = V_0\left[1 + \left(\frac{\phi}{\mu}\right)^p\right].$$

V. TOPOLOGICAL DEFECTS

The toy model is the **discrete Goldstone model**, which has a **real scalar field** in a **mexican hat potential**;

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi),$$
$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2.$$

This potential is $\mathbb{Z}_2$ symmetric:

$$V(g\phi) = V(\phi), \quad g \in \mathbb{Z}_2.$$

If we **expand about the minimum**, then

$$\phi = \pm\eta + \psi \quad \Rightarrow \quad V(\psi) = \lambda\eta^2\psi^2 + \mathcal{O}(\psi^3).$$

That is, we pick up a **massive Goldston boson**, $m_\psi = \sqrt{2\lambda\eta^2}$, and loose the $\mathbb{Z}_2$ symmetry. The mass m^2 is is the coefficient of $\frac{1}{2}\phi^2$.

Therefore, we say **upon choosing the vacuum, the symmetry is spontaneously broken**, with the release of a massive Goldstone boson.

The **effective potential** is that experienced by a particle when coupled to a heat bath,

$$V_{\text{eff}} = V + f_n,$$

where f_n is the free-energy and V the "bare" potential. For bosons, we have

$$f = \frac{m^2}{24}T^2 - \frac{\pi^2}{90}T^4, \quad m^2 = \frac{d^2V}{d\phi^2} = \lambda\left(3\phi^2 - \eta^2\right);$$

hence, using the discrete Goldstone mexican hat as the bare potential,

$$V_{\text{eff}} = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 + \frac{\lambda}{24}\left(3\phi^2 - \eta^2\right)T^2 - \frac{\pi^2}{90}T^4.$$

If we collect the coefficients of $\frac{1}{2}\phi^2$, we call them the **effective mass**,

$$m_{\text{eff}}^2 = \frac{\lambda}{4}\left(T^2 - T_{\text{c}}\right), \quad T_{\text{c}} = 2\eta.$$

Therefore, doing some stability analysis, we find:

- $T > T_{\text{c}} \Rightarrow$ symmetry restored,

- $T < T_{\rm c} \Rightarrow$ symmetry spontaneously broken.

This is characteristic of a $2^{\rm nd}$-order phase transition, where the position of the minima changes with T.

The **Kibble mechanism** states that initially, given a correlation length $\xi < d_{\rm H} \sim t$, there will be one domain wall of area ξ^2 in a volume ξ^3. Hence, the **initial density of defects** is

$$\rho_{\rm dw,init} \propto \frac{1}{\xi}.$$

The solution to the static equation of motion is the ϕ^4-kink solution,

$$\phi(x) = \eta \tanh\left(\frac{x}{\Delta}\right), \quad \Delta = \left(\frac{2}{\lambda\eta^2}\right)^{1/2};$$

where Δ is the **wall width**. We can use this solution to compute the energy-momentum tensor;

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - g^\mu{}_\nu\mathcal{L} = \frac{\lambda\eta^4}{2}\,\text{sech}^4\left(\frac{x}{\Delta}\right)\text{diag}(1,0,1,1).$$

Thus, taking the 00-component, and integrating over space, gives the **surface energy density**,

$$\sigma = \int_{-\infty}^{\infty} T^0{}_0\,dx = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3.$$

We have a **topologically conserved current**,

$$J^\mu = \varepsilon^{\mu\nu}\partial_\nu\phi \quad \Rightarrow \quad \partial_\mu J^\mu = 0.$$

Associated with this is a **conserved kink number**,

$$N = \frac{1}{2\eta}\int_{-\infty}^{\infty} J^0\,dx, \quad \frac{dN}{dt} = 0.$$

Given the 4-current, this is just

$$N = \frac{\phi(x=+\infty) - \phi(x=-\infty)}{2\eta}.$$

A. General Defect Models

We can discuss general topological defect models. To do so, we define **vacuum manifold** to be the **set of field configurations that give the minimum of the potential**;

$$\mathcal{M} = \{\phi_i : V(\phi_i) = V_0 | V(\phi) > V_0, \forall\phi \neq \phi_i\}.$$

Consider the general **real mexican hat** potential,

$$V(\Phi) = \frac{\lambda}{4}\left(|\Phi|^2 - \eta^2\right)^2, \quad \Phi = \{\phi_i\}_{i=1}^N : \phi_i \in \mathbb{R}\,\forall i.$$

Hence, for this potential, the vacuum manifold is

$$\mathcal{M} \cong S^{N-1}.$$

A Lagrangian composed of such a potential is **invariant under global** $SO(N)$ **transformations of the field**. If we expand about the vacuum manifold, we find massive Goldstone bosons along the non-zero field directions; $m_\psi^2 = 2\lambda\eta^2$; all other bosons are massless.

We classify defects according to their non-trivial homotopy group:

- $\Pi_0(\mathcal{M}) \neq I \Rightarrow$ **domain walls**, where $\sigma \sim \lambda\eta^3$ is the mass per unit area;
- $\Pi_1(\mathcal{M}) \neq I \Rightarrow$ **cosmic strings**, where $\mu \sim \lambda\eta^2$ is the mass per unit length;
- $\Pi_2(\mathcal{M}) \neq I \Rightarrow$ **monopoles**, where $m \sim \eta$ is the mass.

In general, $\Pi_N(S^N) = \mathbb{Z}$, an integer.

Domain walls form when a **discrete group** is broken: $\mathbb{Z}_2 \rightarrow I$.

Cosmic strings form from $G \rightarrow I$ where $G \cong SO(2)$; such as $S = SO(2), U(1)$.

Monopoles form from the creation of $U(1)$: $G \rightarrow H \times U(1)$.