Gravitation:
Quick Guide

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Abstract

This is a quick guide – a summary – of the Gravitation course at the University of Manchester, taught by A.Pilaftsis between Sept '08 and Dec '08. These summary notes are my own work, based upon his lecture notes. A copy of my full lecture notes, on this topic, may be found at `www.jpoffline.com`.

This document very quickly goes through the relevant results of tensor calculus, geodesics, isometries, to build the mathematical framework of curvature and Einstein's general theory of relativity. We then discuss the Schwarzschild solution and FRW Universe.

Keywords: General relativity, tensor calculus, covariant derivatives, curvature, Schwarzschild solution, FRW Universe.

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I. SPECIAL RELATIVITY

The flat Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1),$$

and we use the contravariant position vector

$$x^\mu = (ct, x, y, z).$$

Hence, the covariant position vector is

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -x, -y, -z).$$

The Lorentz transformation matrix is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and corresponds to a boost along the x-axis. This allows us to make a coordinate transformation, such that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

The transformation and inverse are related such that

$$\left(\Lambda^{-1}\right)^\mu{}_\nu \Lambda^\nu{}_\lambda = \delta^\mu_\lambda.$$

This just corresponds to boosting one way, and then back again; as one should expect nothing should happen.

The scalar product is invariant upon Lorentz boost:

$$x_\mu y^\mu = x'_\mu y'^\mu.$$

We use the standard quantities:

- Infinitesimal displacement vectors:

$$dx^\mu = (cdt, d\boldsymbol{x}), \quad dx_\mu = (cdt, -d\boldsymbol{x}).$$

- Line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$

- Proper time:

$$d\tau = \frac{dt}{\gamma}.$$

- 4−velocity and 4−momentum:

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \boldsymbol{u}), \quad p^\mu = mu^\mu = (E/c, \boldsymbol{p}).$$

- The differential operator:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla \right).$$

II. TENSOR FORMALISM

The **weak** equivalence principle is that *gravity couples in the same way to all mass & energy.* The **strong** equivalence principle is that *all laws of physics are the same in an accelerated frame, and in a uniform static gravitational field.*

Gravitational redshift is the effect observed due to photons travelling into or outof a gravitational potential:

$$\frac{d\nu}{\nu} = -\frac{d\phi}{c^2}.$$

As a photon moves out, its wavelength gets redshifted. As a photon moves in, its wavelength gets blueshifted.

A. General Definitions

- A **manifold** is a continuous set of points, which look locally like Minkowski space.
- A manifold endowed with a metric is a **Riemannian manifold**.
- A **curve** is a subset of points within a manifold, such that

$$x^a = x^a(\lambda), \quad \lambda \in \mathbb{R}.$$

  That is, some parameterised curve, with *affine parameter* λ.

- An m-dim **surface** in an n-manifold is such that

$$x^a = x^a(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \in \mathbb{R}.$$

That is, a generalisation of a curve.

B. Mathematical Concepts

A **coordinate transformation** is such that

$$x^\mu \longmapsto x'^\mu = x'^\mu(x^\nu).$$

Infinitesimal displacement vectors transform according to

$$dx'^\mu = J^\mu{}_\nu dx^\nu, \quad dx^\mu = \left(J^{-1}\right)^\mu{}_\nu dx'^\nu,$$

where the **Jacobian** is defined as

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \quad \left(J^{-1}\right)^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}.$$

The Jacobian and inverse-Jacobian relate such that

$$\left(J^{-1}\right)^\nu{}_\mu J^\mu{}_\lambda = \delta^\nu_\lambda.$$

The **tangent** curve, on a manifold, is defined as

$$T^\mu = \frac{dx^\mu}{du},$$

where u is the **affine parameter**.

The **metric** gives a measure of distances on a manifold. One can think about the metric as the quantity which tells dimensions how to mix, upon finding a distance. The **line element** is such that

$$ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu.$$

The line element is invariant under coordinate transformation;

$$ds^2 = ds'^2.$$

Notice that the metric $g_{\mu\nu}$ is a function of position. The metric is symmetric, $g_{\mu\nu} = g_{\nu\mu}$. The metric is a $\binom{0}{2}$-tensor; and thus transforms as

$$g'_{\alpha\beta} = \left(J^{-1}\right)^\mu{}_\alpha \left(J^{-1}\right)^\nu{}_\beta g_{\mu\nu}.$$

The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are related such that

$$g_{\mu\nu}g^{\nu\lambda} = \delta_{\mu}^{\lambda}.$$

Contravariant and **covariant** vectors are defined as

$$A'^{\mu} = J^{\mu}{}_{\nu}A^{\nu}, \quad A'_{\mu} = \left(J^{-1}\right)^{\mu}{}_{\nu} A_{\nu}.$$

A **conformal transformation** is one that preserves the angle θ between two vectors. That is,

$$\cos\theta = \frac{A^{\mu}B_{\mu}}{\sqrt{A^{\nu}A_{\nu}}\sqrt{B^{\nu}B_{\nu}}}$$

is unchanged. A metric transformation

$$\tilde{g}_{\mu\nu} = \Omega(x^{\alpha})g_{\mu\nu}, \quad \Omega(x^{\mu}) \neq 0$$

preserves that angle. Such a metric $\tilde{g}_{\mu\nu}$ is a **conformal metric**.

A **tensor** is a set of objects that transform in a specific way. A few types are:

$$\begin{aligned} F'^{\mu\nu} &= J^{\mu}{}_{\alpha}J^{\nu}{}_{\beta}F^{\alpha\beta}, \\ F'_{\mu\nu} &= \left(J^{-1}\right)^{\alpha}{}_{\mu}\left(J^{-1}\right)^{\beta}{}_{\nu} F_{\alpha\beta}, \\ F'^{\mu}{}_{\nu} &= J^{\mu}{}_{\alpha}\left(J^{-1}\right)^{\beta}{}_{\nu} F^{\alpha}{}_{\beta}. \end{aligned}$$

The tensors above are of type $\binom{2}{0}$, $\binom{0}{2}$ and $\binom{1}{1}$. Notice that there is a Jacobian for each contravariant index, and an inverse-Jacobian for each covariant index. If a tensor has all zero components in one frame, then all of its components in all frames, are zero. A **symmetric** tensor is such that

$$A^{\mu\nu} = A^{\nu\mu}, \quad A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right).$$

An **anti-symmetric** tensor is one that

$$A^{\mu\nu} = -A^{\nu\mu}, \quad A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right).$$

C. Tensor Calculus

When differentiating basis vectors, we say that we relate back to the original according to an expansion, such that

$$\partial_{\nu}\boldsymbol{e}_{\mu} = \Gamma^{\rho}{}_{\nu\mu}\boldsymbol{e}_{\rho}.$$

Using this, and considering $\partial_\nu \boldsymbol{A} = \partial_\nu \left(A^\mu \boldsymbol{e}_\mu\right)$, we end up with the **covariant derivative**

$$\begin{aligned} \nabla_\nu A^\mu &= \partial_\nu A^\mu + \Gamma^\mu{}_{\nu\lambda} A^\lambda \\ \nabla_\nu A_\mu &= \partial_\nu A_\mu - \Gamma^\lambda{}_{\nu\mu} A_\lambda . \end{aligned}$$

Is is easy to show that the covariant derivative of a scalar is just the usual partial derivative:

$$\nabla_\nu \left(A^\mu A_\mu\right) = \nabla_\nu \phi = \partial_\nu \phi .$$

We define the **absolute derivative** of a quantity A^μ as

$$\frac{DA^\mu}{Du} = T^\nu \nabla_\nu A^\mu ,$$

where T^μ is the tangent vector along a curve with affine parameter u.

We can derive the transformation rule of $\Gamma^\lambda{}_{\mu\nu}$ by using $\partial'_\nu \boldsymbol{e}'_\mu = \Gamma'^\rho{}_{\nu\mu} \boldsymbol{e}'_\rho$, transforming the LHS, until we arrive at

$$\Gamma'^\pi{}_{\mu\nu} = J^\pi{}_\lambda \left(J^{-1}\right)^\alpha{}_\mu \left(J^{-1}\right)^\beta{}_\nu \Gamma^\lambda{}_{\alpha\beta} + J^\pi{}_\lambda \left(J^{-1}\right)^\alpha{}_\mu \partial_\alpha \left(J^{-1}\right)^\lambda{}_\nu .$$

Notice that the presence of the second term means that the $\Gamma^\lambda{}_{\mu\nu}$ are not tensors.

A **local inertial frame** (LIF) can be shown to be locally flat (i.e. have constant Minkowski metric). To do so, we make the coordinate transformation

$$x'^\mu = \bar{x}^\mu + \frac{1}{2} \Gamma^\mu{}_{\alpha\beta} \bar{x}^\alpha \bar{x}^\beta , \quad \bar{x}^\mu \equiv x^\mu - x^\mu_* .$$

We use this coordinate transformation to derive the Jacobian and its derivative,

$$J^\mu{}_\nu = \delta^\mu_\nu + \mathcal{O}\left(\bar{x}^\beta\right) , \quad \partial_\alpha J^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha} .$$

We then plug these into the transformation rule of the $\Gamma^\lambda{}_{\mu\nu}$, to find that

$$\Gamma'^\lambda{}_{\mu\nu} = \mathcal{O}\left(\bar{x}^\beta\right) .$$

So, at the point $x^\mu = x^\mu_*$ (i.e. where $\bar{x}^\mu = 0$), we see that $\Gamma'^\lambda{}_{\mu\nu}(x^\rho_*) = 0$. Hence, we see that

$$g_{\mu\nu}\left(x^\rho_*\right) = \eta_{\mu\nu} , \quad \partial_\alpha g_{\mu\nu}\left(x^\rho_*\right) = 0 , \quad \partial^\beta \partial_\alpha g_{\mu\nu}\left(x^\rho_*\right) \neq 0 .$$

Torsion is defined to be the anti-symmetric part

$$T^\lambda{}_{\mu\nu} \equiv \frac{1}{2} \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) ,$$

and can be shown to be a tensor. We shall always use **torsion free space**, whereby the $\Gamma^\lambda{}_{\mu\nu}$ is symmetric in its lower indices;

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} .$$

D. Geodesics

A **geodesic** is a curve that gives the extremal of motion, between two points.

An **affine geodesic** is that curve for which a tangent vector is parallel transported to itself. That is,

$$\frac{DT^\mu}{Du} = \lambda(u)T^\mu,$$

where u is the **affine parameter** along the curve. By definition, this is just

$$T^\nu \nabla_\nu T^\mu = \lambda(u)T^\mu,$$

which can easily be put into the form

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \lambda(u)\dot{x}^\mu,$$

where an over-dot denotes derivative with respect to the affine parameter. If $\lambda(u) = 0$, we say that the geodesic is **affinely parameterised**. Thus, on an affinely parameterised geodesic, $T^\nu \nabla_\nu T^\mu = 0$. We call the $\Gamma^\mu{}_{\alpha\beta}$ the **affine connection**, or just **connection**.

The **metric geodesic** is derived by extremising the action

$$S = \int ds = \int du \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu},$$

where we see that the Lagrangian is just

$$L = \dot{s} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}.$$

To extremise the action, we subject this Lagrangian to the Euler-Lagrange equations,

$$\frac{d}{du}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0.$$

Again, we note that $\dot{z}^\mu \equiv dz^\mu/du$, where u is the affine parameter. One finds that the resulting equation is

$$\ddot{x}^\mu + \left\{{}_\alpha{}^\mu{}_\beta\right\} \dot{x}^\alpha \dot{x}^\beta = 0,$$

where

$$\left\{{}_\alpha{}^\mu{}_\beta\right\} \equiv \frac{1}{2} g^{\mu\lambda} \left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\right).$$

We call the $\left\{{}_\alpha{}^\mu{}_\beta\right\}$ Christofell symbols of the second kind, or, just the **Christofell symbols**. This geodesic is called an **affinely parameterised metric geodesic**.

In a torsion free space, and a space for whom the covariant derivative of the metric is zero, we are able to prove the equivalence of the Christofell symbols and connection. That is,

$$T^{\lambda}{}_{\mu\nu} = 0, \quad \nabla_{\alpha} g_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^{\lambda}{}_{\mu\nu} = \left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\}.$$

We prove this by computing $\nabla_{\alpha} g_{\mu\nu} = 0$, and permuting the indices. We can compute the Christofell symbols by finding the geodesics resulting from $L_{\text{eff}} \equiv L^2$, and reading off the corresponding coefficients.

An **isometry** is a coordinate transformation that leave the metric in the same form. Upon the coordinate transformation

$$x^{\mu} \longmapsto x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu}, \quad \epsilon \ll 1,$$

we derive the Jacobian

$$J^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon \partial_{\nu} \xi^{\mu}.$$

Upon plugging this into the transformation of the metric required,

$$g_{\mu\nu}(x^{\rho}) = J^{\alpha}{}_{\mu} J^{\beta}{}_{\nu} g_{\alpha\beta}(x'^{\rho}),$$

we find **Killing's equation**,

$$\nabla_{\nu} \xi_{\mu} + \nabla_{\mu} \xi_{\nu} = 0.$$

Any vector field ξ^{μ} that satisfies such an equation is called a **Killing vector**. The quantity $T^{\mu} \xi_{\mu}$ is conserved along an affinely parameterised geodesic. That is,

$$\frac{D}{Du} \left(T^{\mu} \xi_{\mu} \right) = 0.$$

As the Lagrangian is related to the line element, we say that it can take on one of three values:

$$L^2 = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = \begin{cases} +1 & \text{time-like,} \\ 0 & \text{null,} \\ -1 & \text{space-like.} \end{cases}$$

III. CURVATURE

The commutator of covariant derivatives, acting upon a scalar is zero;

$$\left[\nabla_{\mu}, \nabla_{\nu} \right] \phi = 0,$$

but on a vector gives the **Ricci identity**

$$[\nabla_\mu, \nabla_\nu] A^\rho = R^\rho{}_{\lambda\mu\nu} A^\lambda.$$

We define the Riemann tensor,

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho{}_{\lambda\nu} - \partial_\nu \Gamma^\rho{}_{\lambda\mu} + \Gamma^\rho{}_{\mu\beta}\Gamma^\beta{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\beta}\Gamma^\beta{}_{\mu\lambda}.$$

In a LIF, the Riemann tensor looks like

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho{}_{\lambda\nu} - \partial_\nu \Gamma^\rho{}_{\lambda\mu},$$

from which we can see the **symmetries** of the tensor;

$$\begin{aligned} R_{\alpha\lambda\mu\nu} &= -R_{\lambda\alpha\mu\nu} \\ &= -R_{\alpha\lambda\nu\mu} \\ &= R_{\mu\nu\alpha\lambda}. \end{aligned}$$

Furthermore, we use these to see that

$$R_{\alpha\lambda\mu\nu} + R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} = 0.$$

If we contract the Riemann tensor on its first and third indices, we get the **Ricci tensor**

$$R_{\mu\nu} = g^{\lambda\alpha} R_{\alpha\mu\lambda\nu} = R^\lambda{}_{\mu\lambda\nu}.$$

If we contract the Ricci tensor, we get the **Ricci scalar**,

$$R = g^{\mu\nu} R_{\nu\mu} = R^\mu{}_\mu.$$

If all components of the Riemann tensor are zero, the space is flat:

$$R^\rho{}_{\lambda\mu\nu} = 0 \quad \Longleftrightarrow \quad \text{flat space.}$$

This isn't necessarily true for either the Ricci tensor or scalar.

The **Bianchi identity** is derived by finding $\nabla_\pi R^\rho{}_{\lambda\mu\nu}$ in a LIF, and permuting the indices; to give

$$\nabla_\pi R_{\mu\nu\rho\lambda} + \nabla_\nu R_{\pi\mu\rho\lambda} + \nabla_\mu R_{\nu\pi\rho\lambda} = 0.$$

Contracting the Bianchi identity results in the **contracted Bianchi identity**

$$\nabla^{\mu} G_{\mu\nu} = 0,$$

where the **Einstein tensor** is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

The Einstein tensor is symmetric, $G_{\mu\nu} = G_{\nu\mu}$.

The **geodesic deviation**

$$\frac{d^2 \delta^{\mu}}{du^2} = R^{\mu}{}_{\alpha\beta\rho} T^{\alpha} T^{\beta} \delta^{\rho}$$

describes the relative motion of two affinely parameterised metric geodesics, for two objects freely falling.

IV. ENERGY-MOMENTUM TENSOR

The $T^{\mu\nu}$ are defined to be **flux of** p^{μ} **through the hypersurface** $x^{\nu} =$ **const**. Furthermore, the energy-momentum tensor has the structure

$$T^{\mu\nu} = \begin{pmatrix} \text{energy density} & \text{energy flux} \\ \text{momentum density} & \text{stress tensor} \end{pmatrix}.$$

We tend to write $T^{00} = T^{tt} = \varepsilon$ and $T^{i0} = T^{it} = \pi^i$ as the energy and momentum density. Furthermore, the tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

The **conservation equation** is

$$\nabla_{\nu} T^{\mu\nu} = 0.$$

In a LIF, this is just $\partial_{\nu} T^{\mu\nu} = 0$. If we take $\mu = 0$, then we arrive at the **continuity equation**

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \boldsymbol{\pi} = 0.$$

If we take $\mu = i$ we arrive at

$$\frac{\partial \pi^i}{\partial t} + \frac{\partial T^{ij}}{\partial x^j} = 0,$$

which, using $\phi^i \equiv -\frac{\partial T^{ij}}{\partial x^j}$, is just

$$\frac{\partial \pi^i}{\partial t} = \phi_i,$$

the statement that force is the rate of change of momentum. That is, Newton's law of motion.

A **perfect fluid** is one for whom there is **no heat conduction** or **viscosity**. These conditions mean that $T^{it} = 0$ and $T^{ij} = \delta^{ij}T^{ij}$ respectively. This leaves the energy-momentum tensor diagonal. Therefore, for a perfect fluid in its **comoving frame**, the energy-momentum tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix},$$

where P is the pressure of the fluid. The general form of the energy-momentum tensor is

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu - Pg^{\mu\nu}.$$

Non-relativistic fluids have $P = 0$. For such perfect pressure-less fluids, the continuity equation becomes $\partial_\mu \left(\rho u^\mu u^0\right) = 0$, or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

V. THE GENERAL THEORY OF RELATIVITY

As Einstein's vision was that gravity was described by the curvature of spacetime, and that the thing doing the curving is a distribution of mass-energy; he proposed that

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

The LHS of this equation has information about the geometry, in the form of differentials of the metric. The RHS has the distribution of energy and momentum. Notice that by the continuity equation and the contracted Bianchi identity, this equation is consistent upon contraction with the covariant derivative.

The completely equivalent forms of this equation are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu},$$
$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

where $T = T^{\mu}{}_{\mu}$, the trace of the energy-momentum tensor. These are the field equations of general relativity.

We can also add on the **cosmological constant**,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + g_{\mu\nu}\Lambda,$$

which changes nothing, because $G_{\mu\nu}$ is symmetric, and $\nabla_\rho g_{\mu\nu} = 0$.

A. Newtonian Limit

The Newtonian limit of the theory can be found by letting

$$\frac{dx^i}{d\tau} \ll 1,$$

and perturbing the metric by a small amount, about Minkowski;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1.$$

We further require that $h_{\mu\nu}(x^i)$ only. This defines the **weak field limit** of the theory. Doing this, we end up with

$$\Gamma^i{}_{00} = \frac{1}{2}\partial_i h_{00}, \quad \Gamma^0{}_{00} = 0.$$

The geodesic equation reduces to

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}\partial_i h_{00} = 0.$$

Upon comparison with the Newtonian gravitational field equation

$$\frac{d^2x^i}{dt^2} = -\partial_i \Phi,$$

we read off that

$$h_{00} = 2\Phi \quad \Rightarrow \quad g_{00} = 1 + 2\Phi;$$

where Φ is the Newtonian gravitational potential.

In a LIF,

$$R_{00} = \frac{1}{2}\nabla^2 h_{00} = \nabla^2 \Phi,$$

which is just Poisson's equation.

The Newtonian limit of the general theory allows us to deduce the prefactor $8\pi G$ of the field equations.

B. Gravitational Radiation

Upon linearising the general theory, for small perturbations about Minkowski, and under the **Einstein gauge** $\partial_\mu \bar{h}^{\mu\nu} = 0$, we are able to derive

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu},$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h.$$

This wave equation describes gravitational waves, with source in energy-momentum.

VI. SCHWARZSCHILD SOLUTION

We seek a **spherically symmetric** solution (i.e. metric) to the **Einstein equation in vacuum**. That is, $R_{\mu\nu} = 0$. We use a spherically symmetric ansatz

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

and we find that

$$e^\nu = 1 - \frac{r_s}{r}, \quad e^\lambda = \left(1 - \frac{r_s}{r}\right)^{-1},$$

where the **Schwarzschild radius**, or **event horizon**, is $r_s = 2GM$. The mass of the body doing the curving is M. Hence, the **Schwarzschild metric** is

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Notice that as $r \to \infty$, the metric goes to the flat Minkowski metric. Similarly, notice that the metric has Killing vectors associated with t and ϕ. This corresponds to the statement of conservation of energy and of angular momentum.

By taking radial slices, at constant time, one can find

$$\frac{\nu_1}{\nu_2} = \sqrt{\frac{g_{00}(2)}{g_{00}(1)}} = 1 + \Phi(2) - \Phi(1),$$

which is **gravitational redshift**, but derived properly in a curved spacetime.

A. Dynamics

Computing the effective Lagrangian, $L_{\text{eff}} = L^2$, we find

$$L_{\text{eff}} = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right).$$

The t first-integral of this L_{eff} is

$$\left(1 - \frac{r_s}{r}\right)\dot{t} = \text{const} = \varepsilon,$$

which is the statement of constant energy density. The ϕ first-integral of this L_{eff} is

$$r^2\sin^2\theta\dot{\phi} = \text{const} = \ell,$$

which is the statement of constant angular momentum density. If we put this ε and ℓ into L_{eff} we get

$$K = \left(1 - \frac{r_s}{r}\right)^{-1}\left(\varepsilon^2 - \dot{r}^2\right) - r^2\left(\dot{\theta}^2 + \frac{\ell^2}{r^4\sin^2\theta}\right),$$

where

$$L^2 = K = \begin{cases} +1 & \text{time-like,} \\ 0 & \text{null,} \\ -1 & \text{space-like.} \end{cases}$$

Taking constant $\theta = \pi/2$ results in

$$K = \left(1 - \frac{r_s}{r}\right)^{-1}\left(\varepsilon^2 - \dot{r}^2\right) - \frac{\ell^2}{r^2}.$$

This rearranges into

$$\frac{1}{2}\dot{r}^2 = \frac{\varepsilon^2 - K}{2} - \left[\frac{\ell^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{Kr_s}{2r}\right].$$

This is of the form "kinetic energy is total energy minus potential energy". Hence, we write that the **effective potential** is

$$V_{\text{eff}} = \frac{\ell^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{Kr_s}{2r}.$$

The radii of circular orbits are found by solving

$$\frac{dV_{\text{eff}}}{dr} = 0 \quad \Rightarrow \quad r = r_*,$$

and stability checked via

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_*} > 0 \quad \Longleftrightarrow \quad \text{stable.}$$

A circular orbit is one for whom $r = \text{const}$. Particle orbits have $K = 1$, photon orbits have $K = 0$.

Dividing the two first integrals above, one can derive

$$\frac{\ell}{\varepsilon} = \frac{r^2 \dot{\phi}}{\left(1 - \frac{r_s}{r}\right) \dot{t}} \approx r^2 \frac{d\phi}{dt},$$

after using $\theta = \pi/2$.

Light deflection is the effect that a gravitating mass M has upon a photon with impact parameter d. Note that for photons $K = 0$. We can compute the deviation of the photon, from a straight trajectory;

$$\delta\phi_{\text{defl}} = \frac{2r_s}{d}.$$

Perihelion precession is the effect that a gravitating mass M has upon an orbiting body (i.e. particle so $K = 1$). Orbits are complete, but get shifted in starting point; the shift is derived to be

$$\delta\phi_{\text{prec}} = \frac{3\pi r_s^2}{2\ell^2}.$$

Both of these effects have been very successful as **experimental tests of general relativity**.

B. Black Holes

Note that the Schwarzschild metric has **two singularities** at $r = r_s, 0$. Any true singularities would be present in a frame invariant quantity. One can compute

$$R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} = \frac{6r_s^2}{r^6},$$

from which we see that $r = r_s$ is a removable singularity (i.e. only exists due to a choice in coordinate system), and $r = 0$ is an essential singularity.

Spherically symmetric **null geodesics** are derived easily, by noting that $ds^2 = 0, K = 0$. Thus, metric reads

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{r_s}{r}\right)^2.$$

Integrating this results in

$$t = \text{const} \pm r_*,$$

where the **tortoise coordinate** is defined

$$r_* = r + r_s \ln\left|\frac{r}{r_s} - 1\right|.$$

We let the constants be such that we define **Eddington-Finklestein coordinates**

$$u = t - r_*, \quad v = t + r_*.$$

Notice that

$$\frac{dr_*}{dr} = \frac{r}{r - r_s},$$

so that the metric reads

$$ds^2 = \left(1 - \frac{r_s}{r}\right)\left[dt^2 - dr_*^2\right].$$

Therefore, using this choice of coordinate, the metric is not singular at $r = r_s$, but is singular at $r = 0$.

We can note various things from this metric:

- As $r \to r_s$ (from infinity), light cones get sharper;
- As $r < r_s$, light cones point towards the origin, so that particles can never escape.

Hence, we see that the surface $r = r_s$ defines some one-way membrane, the **event horizon**.

VII. FRW UNIVERSE

Based upon the **cosmological principle** that the universe is **homogenous and isotropic**.

The **FRW metric**, based upon this principle, is

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right].$$

The **curvature constant** k aids us in seeing that

$$r = \left\{\begin{matrix} \sin\chi \\ \chi \\ \sinh\chi \end{matrix}\right\} \iff k = \left\{\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right\} \iff \text{geometry} = \left\{\begin{matrix} \text{closed} \\ \text{flat} \\ \text{open} \end{matrix}\right\}.$$

$a(t)$ is the **scale factor**.

Weyl's polstulate is that the **universe is a perfect fluid**. Hence, the energy-momentum tensor for the universe is

$$T_{\mu\nu} = \text{diag}\left(\rho, -P, -P, -P\right), \quad T = \rho - 3P.$$

We can use the non-zero components of the Ricci tensor, derived from the metric, to compute Einstein's equation. The 00-component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right),$$

which is known as the **acceleration equation**, or **Raychaudhuri's equation**. An over-dot here denotes derivative with respect to coordinate time t. Inserting this into the ij-components of Einstein's equation results in

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

known as the **Friedmann equation**. Using the continuity equation $\nabla_\mu T^\mu{}_\nu = 0$, and the Christofell symbols derived from the metric, one can derive the **fluid equation**

$$\dot{\rho} = -3H\left(\rho + P\right).$$

The **Hubble parameter** is defined to be

$$H = \frac{\dot{a}}{a}.$$

The fluid equation can be solved to give dependance of ρ upon a, once an equation of state is known.

$$\begin{aligned}
\text{matter} \quad &\Rightarrow \quad P_m = 0 \quad \Rightarrow \quad \rho_m = \rho_{m,0}a^{-3}, \\
\text{radiation} \quad &\Rightarrow \quad P_r = \tfrac{1}{3}\rho_r \quad \Rightarrow \quad \rho_r = \rho_{r,0}a^{-4}, \\
\text{vacuum} \quad &\Rightarrow \quad P_V = -\rho_V \quad \Rightarrow \quad \rho_V = \text{const.}
\end{aligned}$$

The **critical density** is defined to be that density which gives $k = 0$ in the Friedmann equation;

$$\rho_c = \frac{3H^2}{8\pi G}.$$

Using this, we define density fractions of the various species;

$$\Omega_i = \frac{\rho_{i,0}}{\rho_c}, \quad \Omega_k = -\frac{k}{H_0^2 a_0^2}, \quad i \in \{r, m, V\}.$$

Also,

$$\Omega_m + \Omega_r + \Omega_V + \Omega_k = 1.$$

We then use the Friedmann equation to derive that

$$\frac{H^2}{H_0^2} = \frac{1}{\rho_c}\left(\frac{\rho_{m,0}}{a^3} + \frac{\rho_{r,0}}{a^4} + \rho_{V,0}\right) + \frac{\Omega_k}{a^2},$$

which rearranges to give

$$\frac{\dot{a}}{a} = H_0\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_V + \frac{\Omega_k}{a^2}\right)^{1/2},$$

or, alternatively,

$$\dot{a} = H_0\left(\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + a^2\Omega_V + \Omega_k\right)^{1/2}.$$

Notice that in a **matter dominated** universe, $\Omega_m = 1$, then

$$\dot{a} = H_0 a^{-1/2} \quad \Rightarrow \quad a \propto t^{2/3}.$$

In a **vacuum dominated** universe, $\Omega_V = 1$, then

$$\text{vacuum} \quad \Longleftrightarrow \quad a \propto e^t,$$

which is exponential expansion. In a **radiation dominated** universe, $\Omega_r = 1$, then

$$\text{radiation} \quad \Longleftrightarrow \quad a \propto t^{1/2}.$$

The **age of the universe** is computed via

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}}.$$

In a **matter dominated** universe this reduces to

$$t_0 = \frac{1}{H_0}\int_0^1 da\, a^{1/2} = \frac{2}{3H_0}.$$

Introducing **conformal time** $dt = a d\eta$, the metric is conformal. For a mixed universe, $\Omega_k + \Omega_m = 1$, we can derive

$$\begin{aligned} a(\eta) &= \frac{\Omega_m}{2(\Omega_m - 1)}\left[\cosh\left(\eta H_0\sqrt{1-\Omega_m}\right) - 1\right], \quad \Omega_k > 0, \\ a(\eta) &= \frac{\Omega_m}{2(\Omega_m - 1)}\left[1 - \cos\left(\eta H_0\sqrt{\Omega_m - 1}\right)\right], \quad \Omega_k < 0. \end{aligned}$$

The second expression clearly gives an **oscillatory universe**, with period

$$\eta_{\text{tot}} = \frac{2\pi}{H_0\sqrt{\Omega_m - 1}}.$$

The corresponding total time is

$$t_{\text{tot}} = \int_0^{\eta_{\text{tot}}} d\eta\, a(\eta) = \frac{\pi\Omega_m}{(\Omega_m - 1)^{3/2}H_0}.$$

Thus, we see that a small non-zero k is enough to decide the fate of the universe.

A. Cosmological Redshift

The flat, null FRW metric, ignoring angular dependance, gives

$$dt = a(t)dr.$$

Hence,

$$R = \int_{t_o}^{t_e} \frac{dt}{a(t)} = \int_{t_o+\delta t_o}^{t_e+\delta t_e} \frac{dt}{a(t)},$$

which gives

$$\frac{\delta t_o}{\delta t_e} = \frac{a(t_o)}{a(t_e)}.$$

Hence, for photons emitted at t_e, and observed at t_o, we have

$$\frac{\nu_e}{\nu_o} = \frac{a(t_o)}{a(t_e)} = 1 + z.$$

This is cosmological redshift, due to the **expansion of the universe**.

For small scale factors, we can derive **Hubble's law** $v = HR$.

B. Flatness Problem

We can find that

$$\frac{\Omega_k(t_0)}{\Omega_k(t_r)} = \frac{t_0}{t_r} \approx 10^{60}.$$

That is, the universe used to be a lot flatter than it is now; the degree to which is must have been flat is so small that it is called the **fine tuning problem**. We can introduce the idea of a vacuum dominated epoch, to provide exponential expansion. This is **inflation**, and we find that

$$\frac{\Omega_k(t_r)}{\Omega_k(t_i)} = \frac{a_i^2}{a_r^2},$$

from which, via $a \sim e^t$, we read off that about 60 e-folds are required.