Maths Methods - A Summary

1 Linear Algebra

Some vector space V, over some scalar space S. Let us have some elements of each set

$$a, b, c \in V \qquad \mu, \lambda \in S.$$

Then, we have that addition (of vectors) and multiplication (by scalars) are closed: $a + b \in V$ and $\mu a \in V$.
Addition in commutative: $a + b = b + a$, associative: $(a + b) + c = a + (b + c)$. There is some null element $0 \in V$, such that $a + 0 = a$. Each element has an inverse $(-a) \in V$, such that $a + (-a) = 0$, the null element.

Multiplication is distributive: $\lambda(a + b) = \lambda a + \lambda b$ and $(\lambda + \mu)a = \lambda a + \mu a$. It is also associative: $\lambda(\mu a) = (\lambda\mu)a$. It has a unit element (null), so that $1a = a$.

Vectors $a, b, \ldots, u$ are *linearly independent* if

$$\lambda a + \mu b + \ldots + \sigma u = 0, \tag{1.1}$$

has only one solution, which is that the coefficients are zero.

Functions map from the *domain* into the *codomain*, the subset of the codomain of which the function can 'reach' is the *range*. The range and codomain are not necessarily exactly the same.

1.1 Function Spaces

The space of all functions where

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty, \tag{1.2}$$

is called $\mathcal{L}^2(\mathbb{R})$. So, functions mapping $\mathbb{R} \mapsto \mathbb{C}$.

Any vector in some space can be expanded in terms of basis vectors in the space; that is:

$$\boldsymbol{x} = \sum_i x_i \boldsymbol{e}_i$$

We use the following notation for the *scalar product*: $(\boldsymbol{a}, \boldsymbol{b})$, and it satisfies the following conditions:

$$(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{a})^* \qquad ||\boldsymbol{a}||^{1/2} = (\boldsymbol{a}, \boldsymbol{a}) \geq 0. \tag{1.3}$$

If $(\boldsymbol{a}, \boldsymbol{b}) = 0$, the then two vectors are orthogonal.

For a *discrete basis*, orthnormality and completeness are stated thus:

$$(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} \qquad \sum_i (\boldsymbol{e}_i)_a (\boldsymbol{e}_i)_b = \delta_{ab}. \tag{1.4}$$

Where in the final expression for completness, we used the notation $(\boldsymbol{e}_i)_a$ is the a^{th} component of basis vector $\boldsymbol{e}_i$.

For a *continuous basis*, we have the orthonormality and completeness relations:

$$\int_{-\infty}^{\infty} \phi_k^*(x)\phi_{k'}(x)dx = \delta(k-k') \qquad \int_{-\infty}^{\infty} \phi_k^*(x)\phi_k(x)dx = \delta(x-x') \tag{1.5}$$

The function-version of the scalar product is:

$$(\phi, \psi) = \int \phi^*(x)\psi(x)dx \tag{1.6}$$

We can demonstrate how to decompose a function $\psi(x)$, into a linear combination of orthonormal basis states $\phi_n(x)$. So, the composition and orthonormality relations are:

$$\psi(x) = \sum_n c_n\phi_n(x) \qquad (\phi_m, \phi_n) = \int \phi_m^*(x)\phi_n(x)dx = \delta_{nm} \tag{1.7}$$

Then, we can find the coefficients c_n by multiplying the composition relation with 'another function', and integrate:

$$\int \phi_m^*\psi dx = \sum_n c_n \int \phi_m^*\phi_n dx = \sum_n c_n\delta_{nm} = c_m$$

That is:

$$c_n = (\phi_n, \psi) \tag{1.8}$$

Then, if we use this coefficient in the composition relation, we can show that there are no more coefficients (thus completeness). That is, let us:

$$\psi(x) = \sum_n (\phi_n(x'), \psi(x'))\phi_n(x) = \sum_n \int \phi_n^*(x')\psi(x')\phi_n(x)dx'$$

Then, let us rearrange this:

$$\psi(x) = \int \sum_n \phi_n^*(x')\phi_n(x)\psi(x')dx' = \int \delta(x-x')\psi(x')dx'$$

Which leads us to the delta function.

1.1.1 Delta Function

We define it:

$$\int_{-\infty}^{\infty} f(x')\delta(x-x')dx' = f(x) \tag{1.9}$$

We have the delta function of some constant function:

$$\delta(a(x-x')) = \frac{1}{|a|}\delta(x-x') \tag{1.10}$$

Which we derive by shifting the original variable. We have the delta-function of an arbitrary function:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x')}{g'(x_i)} \tag{1.11}$$

Where x_i is a zero of the function $g(x)$. This is derived by taking a Taylor expansion of $g(x)$ around a zero.

1.2 Linear Operators

An operator $\mathcal{L}$ is linear if the following hold, for some scalars λ, μ:

$$\mathcal{L}(\lambda \boldsymbol{a} + \mu \boldsymbol{a}) = \lambda \mathcal{L}\boldsymbol{a} + \mu \mathcal{L}\boldsymbol{b}$$

The matrix representation of an operator is just:

$$\mathcal{L}_{ji} = (\boldsymbol{u}_j, \mathcal{L}\boldsymbol{e}_i) = \int u_j^* \mathcal{L} e_i \, d\tau \tag{1.12}$$

If an operator maps from the domian, which has a basis $\boldsymbol{e}_i$, to codomain, having basis $\boldsymbol{u}_j$.

The *Hermitian conjugate* (the same as the *adjoint*) of an operator, is such that:

$$M_{ij}^\dagger = M_{ji}^* \qquad (a, Mb) = (M^\dagger a, b) \tag{1.13}$$

An Hermitian (or equivalently, self-adjoint) is one for which $M^\dagger = M$.

1.2.1 Generalised Eigenvalue Problem

Is of the form:

$$\mathcal{L}y(x) = \lambda \rho(x) y(x) \tag{1.14}$$

Where $\rho(x) > 0$ (and real) is the *weight function*. We read the above equation as: the operator $\mathcal{L}$ has eigenstates $y(x)$, with eigenvalue λ, with respect to some weight function $\rho(x)$.

Two different eigenstates u, v are orthogonal, with respect to the weight function, if:

$$(u, v)_\rho = \int \rho(x) u^*(x) v(x) dx = 0 \tag{1.15}$$

2 Sturm-Lioville

Such an operator, for the above generalised eigenvalue equation, is of the form:

$$\mathcal{L} = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\cdot\right) + q(x)\cdot \tag{2.1}$$

Thus, the generalised eigenvalue equatino can be written, using primes to denote derivative:

$$-[py']' + qy = \lambda\rho y$$

So that, everything inserted & the differential expanded out, it looks like:

$$p(x)\frac{d^2y}{dx^2} + \frac{dp}{dx}\frac{dy}{dx} - q(x)y(x) + \lambda\rho(x)y(x) = 0 \tag{2.2}$$

Let me write this more compactly:

$$py'' + p'y' - qy + \lambda\rho y = 0 \tag{2.3}$$

We can put some general DE, as below, we can put it into the above SL form:

$$y'' + \alpha y' + \beta y + \lambda\tau y = 0$$

Where everything, except λ, is a function of x. To put the above DE into SL form: we see that if we multiply it by p, we have something that looks like the SL equation. So, we do so, and look at the factors infront of the y-terms:

$$p' = p\alpha \qquad q = -p\beta \qquad \rho = p\tau$$

Hence, we can solve for p via:

$$p = e^{\int \alpha\, dx} \tag{2.4}$$

Thus, if we have some DE, with unit coefficient of y'', if we multiply it by p, which is computed above, we have an equation in SL form. We also see that as $p > 0$ and $\rho > 0$, then $\tau > 0$.

Now, suppose that $u(x), v(x)$; where $x \in [a, b]$, are eigenstates of some operator of SL form. Then, the SL operator $\mathcal{L}$ is defined on $[a, b]$.

Then, we may construct: $v\mathcal{L}u - u\mathcal{L}v$. If we insert these eigenstates into the SL DE, we can fairly easily get:

$$v\mathcal{L}u - u\mathcal{L}v = [-vpu' + upv']' = [p(uv' - vu')]'$$

Trivially integrate this to:

$$\int_a^b v\mathcal{L}u - u\mathcal{L}v\, dx = [p(uv' - vu')]_a^b$$

Now, if some operator $\mathcal{L}$ is Hermitian, then the following holds:

$$(v, \mathcal{L}u) = (\mathcal{L}v, u)$$

So (we have assumed that the eigenstates are real), we therefore have a condition for a SL operator to be Hermitian:

$$[p(uv' - vu')]_a^b = 0 \tag{2.5}$$

That is, on the range $[a, b]$, $\mathcal{L}$ is an Hermitian SL operator. Hence, just evaluating this:

$$p(b)u(b)v'(b) - p(a)v(a)u'(a) = 0$$

The condition for a SL operator to be Hermitian. Now, if $p(a) = p(b) = 0$, then a, b are denoted *singular points*.

A useful way to remember the ordering of $uv' - vu'$ above, is to think of a determinant: it is called the Wronskian.

2.1 Second Solutions

For any 2nd order DE, there are always 2 solutions. So, we will have:

$$\mathcal{L}u = \rho\lambda u \qquad \mathcal{L}v = \rho\lambda v$$

Thus, multiplying each by the other solution:

$$v\mathcal{L}u = v\rho\lambda u \qquad u\mathcal{L}v = u\rho\lambda v$$

Thus, subtracting the second from the first:

$$v\mathcal{L}u - u\mathcal{L}v = \lambda\rho(vu - uv) = 0$$

However, we also have the relation that:

$$v\mathcal{L}u - u\mathcal{L}v = [p(uv' - vu')]'$$

Hence, its RHS is equal to zero. Hence, the follwing must be true:

$$p(uv' - vu') = c$$

Where c is some constant. We can then find the 'second solution', v, from this.
We have that $uv' - vu' = c/p$. Then, suppose that $v = uw$, for some function w. Then, $v' = u'w + uw'$. Hence, after a little rearranging, we have:

$$w' = \frac{a}{u^2 p}$$

Thus, we have our second solution via:

$$w(x) = \int \frac{c}{u^2(x)p(x)}dx \qquad v(x) = u(x)w(x) \tag{2.6}$$

However, we also have that $p(a) = p(b) = 0$. Hence, $\mathcal{L}$ is non-Hermitian for second-solutions.

A Hermitian Sl operator have real eigenvalues, and eigenstates are non-degenerate.

3 Series Solutions

This section describes the way one finds a recurrence relation for some DE. The DE we use for this example is:

$$f''(x) - 2xf'(x) + (2\lambda - 1)f(x) = 0$$

A *series solution* is where we assume a solution of the form:

$$f(x) = \sum_n c_n x^n \tag{3.1}$$

Then, the problem is to find the coefficients c_n. To do this, we derive a recurrence relation that links them. Let us compute:

$$f'(x) = c_n n x^n \quad f''(x) = c_n n(n-1)x^{n-1}$$

So, putting these expressions into the DE:

$$c_n n(n-1)x^{n-2} - 2c_n n x^n + (2\lambda - 1)c_n x^n = 0$$

Now comes standard 'trick'. We shift the summation variable of one (or more) expressions, so that all expressions have the same power of x. Giving:

$$c_{n+2}(n+2)(n+1)x^n - 2c_n n x^n + (2\lambda - 1)c_n x^n = 0$$

Hence, by linear independance, we rearange to get:

$$c_{n+2} = \frac{2n - (2\lambda - 1)}{(n+2)(n+1)} c_n \tag{3.2}$$

Thus a recurrence relation found.

3.1 Legendre Polynomials

The Legendre DE is:

$$[(1-x^2)y'(x)]' + \lambda y(x) = 0$$

Again, we find a series solution, $y(x) = \sum_n c_n x^n$; to be:

$$c_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} c_n \tag{3.3}$$

The Legendre polynomials have a *generating function*:

$$\sum_n t^n P_n(x) = f(x,t) = \frac{1}{\sqrt{1 - 2tx + t^2}} \qquad t \in [-1, 1] \tag{3.4}$$

4 Green Functions

If we want to solve some DE:

$$\mathcal{L}u(x) = f(x)$$

Then, if we let:

$$\mathcal{L}G(x, x') = \delta(x - x') \tag{4.1}$$

Then, the solution is just the convolution of the driving force with the Green function; thus:

$$u(x) = \int G(x, x') f(x') dx' \tag{4.2}$$

Making sure that it is understood that $\mathcal{L}$ operates on x only (i.e. not x'); and that boundary conditions must be specified. In general, a different Green function $G(x, x')$ will be found for different boundary conditions.

So, once a Green function is found; which is found by solving the homogeneous version of the DE of interest; we can find the solution to the DE with any driving force, for the same boundary conditions.

There are 3 methods for finding Green functions: eigenstate method, continuity method & Fourier transform method.

4.1 Eigenstate Method

If $u_n(x)$ are a complete set of orthonormal eigenstates of $\mathcal{L}$; where:

$$\mathcal{L}u_n(x) = \lambda_n u_n(x)$$

Then the Green function is given by:

$$G(x, x') = cu_0(x) + \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x) u_n^*(x') \tag{4.3}$$

If $\lambda_0 = 0$. If $\lambda_0 \neq 0$, then the sum can be taken from zero, and the first term ignored.

4.2 Continuity Method

This is only for 1D DE's:

$$\left[\frac{d^2}{dx^2} + \alpha(x)\frac{d}{dx} + \beta(x)\right] G(x, x') = \delta(x - x')$$

So, we solve for $x < x', x > x'$ separately (not the equation becomes homogeneous). Then, solve discontinuity at $x = x'$:

$$\frac{d}{dx}G(x' + \epsilon, x') - \frac{d}{dx}G(x' - \epsilon, x') = 1 \tag{4.4}$$

4.3 Fourier Transform Method

Use for time dependent wave equations:

$$\mathcal{L}_{\boldsymbol{x},t} G(\boldsymbol{x}, t) = \delta(\boldsymbol{x})\delta(t)$$

We use the following Fourier transforms:

$$\tilde{G}(\boldsymbol{k}, \omega) = \int d^3r \int dt\, G(\boldsymbol{x}, t) e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{4.5}$$

$$G(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \tilde{G}(\boldsymbol{k}, \omega) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{4.6}$$

With the delta-functions being the FT of unity:

$$\delta(\boldsymbol{x})\delta(t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{4.7}$$

So, to find the Green function $G(\boldsymbol{x}, t)$, put the above G and delta functions into the DE. Then, the operator $\mathcal{L}$ will be inside the integral. Operate with $\mathcal{L}$ on the inside G. Then, upon comparison of integrands, one will find:

$$\tilde{G}(\boldsymbol{k}, \omega)\mathcal{L}e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} = 1 \tag{4.8}$$

Thus, we have found $\tilde{G}(\boldsymbol{k}, \omega)$. Then, put it back into the FT expression for $G(\boldsymbol{x}, t)$. Hence, the Green function has been found, once the integral performed.

4.3.1 Contour Integration

We will probably have to use contour integration to evaluate the ω integral:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t}$$

It will probably turn out that $f(\omega)$ has simple poles on the real axis. That is, as one integrates over the real axis, a singularity will be passed over. Usually, for such problems, we have the causality b.c. that $t > 0$. Hence, we will be using residues in the lower-half-plane. Suppose:

$$\int_{-\infty}^{\infty} \frac{1}{\omega - x} e^{-i\omega t} \frac{d\omega}{2\pi}$$

Then the residue(s) are:

$$e^{-i\omega t} 2x$$

And the integral is $(-2\pi i)$ times the sum of such residues.