Electrodynamics - A Summary

1 Preliminaries

Vector relations:

$$\begin{aligned} \nabla \cdot \nabla \times \boldsymbol{v} &= 0 & (1.1) \\ \nabla \times \nabla \times \boldsymbol{v} &= \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v} & (1.2) \end{aligned}$$

Divergence & Stokes Theorem:

$$\begin{aligned} \int \boldsymbol{v} \cdot d\boldsymbol{S} &= \int \nabla \cdot \boldsymbol{v} dV & (1.3) \\ \oint \boldsymbol{v} \cdot d\boldsymbol{\ell} &= \int \nabla \times \boldsymbol{v} \cdot d\boldsymbol{S} & (1.4) \end{aligned}$$

Delta-function; think about it as filtering out a single value of a function. Use them in representing point-charges:

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x - x')dx' \tag{1.5}$$

Operators: If the following are used to operate on a plane wave, then we find the following results:

$$\frac{\partial}{\partial t} \to -i\omega \qquad \nabla\times \to i\boldsymbol{k}\times \tag{1.6}$$

Maxwells equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0} & (1.7) \\ \nabla \cdot \boldsymbol{B} &= 0 & (1.8) \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} & (1.9) \\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} & (1.10) \end{aligned}$$

We have the relation:

$$\boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{k}} \times \boldsymbol{E} \tag{1.11}$$

In linear media:

$$\boldsymbol{D} = \varepsilon_r \varepsilon_0 \boldsymbol{E} \qquad \boldsymbol{H} = \frac{1}{\mu_r \mu_0}\boldsymbol{B} \tag{1.12}$$

In vacuum, Maxwells equations may be written:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \nabla \cdot \boldsymbol{D} = \rho \tag{1.13}$$

The Lorentz force law, and Poynting vector:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \qquad \boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H} \tag{1.14}$$

2 Fields 1

A useful way to think about all of these integrals: The distribution is always in the primed coordinate system. The integrals sweep over the distribution, picking out the little bits of charge, then seamlessly adds them together via the integral.

If the observation point is some $\boldsymbol{r}$, then if there is a point charge, at $\boldsymbol{r}'$, the electric field at the observation point is given by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{2.1}$$

The electric (scalar) potential, due to some continuous distribution of charge, residing at primed coordinates, is given by:

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{2.2}$$

The electric field is given by (static fields only):

$$\boldsymbol{E} = -\nabla\phi \tag{2.3}$$

We easily combine this and Gauss' law, to derive:

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{2.4}$$

We have, by analogy, this magnetic vector potential:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{2.5}$$

The magnetic field is related to the vector potential via:

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{2.6}$$

We use this, and the Coulomb gauge (for static fields), $\nabla\cdot\boldsymbol{A} = 0$, to derive:

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J} \tag{2.7}$$

The continuity equation, for charges & currents:

$$\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \tag{2.8}$$

3 Fields in Materials

If we have an interface, region 1 into region 2. At the interface, is some charge distribution σ. Then:

$$D_{2,\perp} - D_{1,\perp} = \sigma \tag{3.1}$$

So, we also have the following relations:

$$\begin{aligned}
\varepsilon_{r,2}\varepsilon_0 E_{2,\perp} - \varepsilon_{r,1}\varepsilon_0 E_{1,\perp} &= \sigma & (3.2)\\
\varepsilon_{r,2}\varepsilon_0\frac{\partial V_2}{\partial r} - \varepsilon_{r,1}\varepsilon_0\frac{\partial V_1}{\partial r} &= -\sigma & (3.3)
\end{aligned}$$

4 Gauges

The Coulomb gauge, used for static fields:

$$\nabla \cdot \boldsymbol{A} = 0 \tag{4.1}$$

The Lorentz gauge, used for time-varying fields:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{4.2}$$

We then have that fields can be found from potentials via:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{4.3}$$

The choices of $\phi, \boldsymbol{A}$ are arbitrary, up to certain factors:

$$\phi' = \phi - \frac{\partial \chi}{\partial t} \qquad \boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \tag{4.4}$$

Under the Lorentz gauge, we are able to derive the following wave-equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = -\mu_0 \boldsymbol{J} \tag{4.5}$$

To do this, put $\boldsymbol{E}$ into Gauss' law, and put both $\boldsymbol{E}, \boldsymbol{B}$ into Amperes law. This will produce two coupled wave equations. To un-couple, use the Lorentz gauge. We end up with:

$$\Box^2\phi = -\frac{\rho}{\varepsilon_0} \qquad \Box^2\boldsymbol{A} = -\mu_0\boldsymbol{J} \qquad \Box^2 \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{4.6}$$

5 Energy in Fields

The energy density in each field is given by:

$$U_E = \varepsilon_0 \int |\boldsymbol{E}|^2 d^3r \tag{5.1}$$

$$U_B = \mu_0 \int |\boldsymbol{B}|^2 d^3r \tag{5.2}$$

The Poynting vector is the energy flow per unit area, per unit time:

$$\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H} \tag{5.3}$$

These above equations can then be bought together into Poyntings theorem, which gives the rate of flow of energy:

$$\frac{dW}{dt} = -\frac{\partial}{\partial t}(U_E + U_M) - \int \boldsymbol{P} \cdot d\boldsymbol{S} \tag{5.4}$$

We will often denote the Poynting vector in terms of the total flux of energy, per unit time:

$$\bar{P} = \int \boldsymbol{P} \cdot d\boldsymbol{S}$$

6 Laplace's Equation

The solution to Laplaces equation $\nabla^2\phi = 0$, in cylindrical polars, are Bessel functions. In Spherical polars, the solution has the form:

$$V(r,\theta,\varphi) = \sum_{\ell m}\left(A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right)Y_{\ell m}(\theta,\varphi)$$

If the system has axial symmetry, then this reduces to (and this is more commonly used):

$$V(r,\theta) == \sum_{\ell m}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{6.1}$$

If there is total angular symmetry to the system, then the solution is of the form:

$$V(r) = A + \frac{B}{r}$$

7 Multipole Expansions

We have:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_\ell \frac{1}{r^{\ell+1}}\int r'^\ell P_\ell(\cos\gamma)\rho(\boldsymbol{r}')d^3r' \tag{7.1}$$

With monopole and dipole moments being:

$$q \equiv \int \rho(\boldsymbol{r}')d^3r' \qquad p = \int r'\cos\gamma\rho(\boldsymbol{r}')d^3r' \tag{7.2}$$

Where $\gamma \equiv \theta' - \theta$. It is also usual to take the vector version of the dipole moment to be:

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}')d^3r'$$

So that the potential due to a dipole is:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\boldsymbol{p}\cdot\hat{\boldsymbol{r}} \tag{7.3}$$

The magnetic dipole moment, which is derived in a very similar way, is:

$$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')d^3r' \tag{7.4}$$

With the magnetic potential due to a magnetic dipole being given by:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$

8 Retarded Fields

The scalar (vector is directly analogous) potential, due to retarded sources, is:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' \tag{8.1}$$

Where:

$$t' = t - \frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'| \tag{8.2}$$

For point charges, we get the Lienard-Wiechert fields:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\kappa_r R_r} \qquad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{qc\boldsymbol{\beta}_r}{\kappa_r R_r} \qquad \kappa \equiv 1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta} \qquad \boldsymbol{R} \equiv \boldsymbol{r}-\boldsymbol{r}' \tag{8.3}$$

We find that the electric (and magnetic) fields can be written as a sum of a velocity and acceleration components:

$$E_v \propto \frac{1}{R^2} \qquad E_a \propto \frac{1}{R}$$

Then, the radiation is:

$$P \propto E^2 \quad \Rightarrow \quad P_v \propto \frac{1}{R^4} \quad P_a \propto \frac{1}{R^2}$$

Hence, the total radiation:

$$\bar{P}_v \propto \frac{1}{R^2} \qquad \bar{P}_a \propto 1$$

In this way, we see that charges moving at constant velocity do not radiate. This is a direct result from the fact that an inertial frame can always be constructed in which a constant-velocity particle is at rest.

Hence, only accelerating charges radiate.