Cosmology - A Summary

1 Preliminaries

Throughout, I shall denote quantities evaluated 'now' by a subscript 0. For example, the Hubble parameter, now, is given by $H(t_0) \equiv H_0$. Notice that t_0 is the time 'now': the time after the big bang.

1pc $\approx$ 3light years.

Redshift:

$$z \equiv \frac{\lambda_0}{\lambda_e} - 1 \tag{1.1}$$

The Cosmological principle: we are not at a special location. The CP is both isotropy (*invariant under rotation*) and homogeneity (*invariant under translation*).

Objects with internal forces do not participate in universe expansion: internal forces of galaxies are stronger than that which acts to rip them apart.

Cosmic time: Time at which all observers observe the same CMBR temperature, or Hubble parameter.

Comoving coordinates:

$$\boldsymbol{r}(t) = a(t)\boldsymbol{x} \tag{1.2}$$

Where $\boldsymbol{r}(t)$ is the *proper distance*, $a(t)$ the *scale factor* and $\boldsymbol{x}$ the *comoving distance*, which is constant.

From $v = Hd$, which is just $\dot{r} = Hr$, we can derive an expression for the Hubble parameter:

$$H(t) = \frac{\dot{a}}{a} \tag{1.3}$$

From this, we can link the scale factor with redshift:

$$1 + z = \frac{a(t_0)}{a(t)} \tag{1.4}$$

Where a subscript 0 denotes 'now'. We usually set $a(t_0) = 1$. That is:

$$1 + z = \frac{1}{a(t)} \tag{1.5}$$

2 Friedmann Equation

We derive this by (essentially) saying that the sum of the kinetic and potential energies is a constant.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2} \tag{2.1}$$

It is a 'snapshot' equation, and gives ρ, the *total relativistic mass density* at a given cosmic time.

$$\rho = \rho_m + \rho_\gamma + \rho_\Lambda$$

Where the total relativistic mass density is given by the sum over all species contributing: matter (dark & baryonic), radiation (photons & neutrinos) and vacuum energy (dark energy)

2.1 Curvature Constant k

- If $k > 0$, then the interpretation is that the potential energy of the universe 'wins', the geometry is spherical, and the universe will 'die' in a big crunch: closed.
- If $k = 0$, then there is a 'tie', and the universe will just come to a halt. The geometry is flat.
- If $k > 0$, then the kinetic energy 'wins', with the universe expanding forever: open. The geometry is hyperbolic.

We observe $k = 0$.

3 Fluid Equation

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\left[\rho + \frac{p}{c^2}\right] = 0 \tag{3.1}$$

We consider *species*: matter having $p = 0$, and radiation having $p_\gamma = \frac{1}{3}\rho_\gamma c^2$.

3.1 Matter Only

Here, we solve the fluid equation, with $\rho \to \rho_m$ and $p = 0$. We can then derive the following dependancies:

$$\rho_m(a) \propto a^{-3} \tag{3.2}$$
$$a_m(t) \propto t^{2/3} \tag{3.3}$$
$$\rho_m(t) \propto t^{-2} \tag{3.4}$$
$$H(t) = \frac{2}{3t} \tag{3.5}$$

We see that density scales as the volume increases (a^3).

3.2 Radiation Only

Here, we solve the fluid equation, with p_γ, as given above; finding:

$$\rho_\gamma(a) \propto a^{-4} \quad (3.6)$$
$$a_\gamma(t) \propto t^{1/2} \quad (3.7)$$
$$\rho_\gamma(t) \propto t^{-3} \quad (3.8)$$
$$H(t) = \frac{1}{2t} \quad (3.9)$$

Here, we see that the density scales as a combination of volume and redshift.

3.3 Mixtures

In a species dominant era, the (time variation) parameters are set by that species.

3.3.1 Early Universe

As we shall see, this is a radiation dominant era. Thus, a scales as $t^{1/2}$. Hence, $\rho_m \propto t^{-3/2}$; and $\rho_\gamma \propto t^{-2}$. The radiation domination falls, and matter becomes dominant:

3.3.2 Later Times

Here, we have matter domination; hence $a \propto t^{2/3}$. Therefore $\rho_\gamma \propto t^{-8/3}$ and $\rho_m \propto t^{-2}$.

4 Observations

4.1 Hubble Parameter

Note, at any time in the past: $H(t) > H_0$. Its latest value is:

$$H_0 = 72.6 \pm 6 kms^{-1} Mpc^{-1}$$

Or, in SI:

$$H_0 = 2.27 \times 10^{-18} s^{-1}$$

4.2 Density Parameter Ω

If $k = 0$, then we can define a critical density (from the FE) for which the universe has the density required to slow it to a halt:

$$\rho_{crit} \equiv \frac{3H^2}{8\pi G} \quad (4.1)$$

In SI units:

$$\rho_{crit} \approx 1 \times 10^{-26} kgm^{-3}$$

Then, we define a density parameter, to see 'how close' we are to this critical case:

$$\Omega \equiv \frac{\rho}{\rho_{crit}} \tag{4.2}$$

We observe $\Omega_0 = 1$.

4.3 Acceleration

By differentiating the FE, and substituting in the fluid equation, we can derive the *acceleration equation*:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{4.3}$$

And further, after Taylor expanding the scale factor (i.e. be careful about using this over 'long-times'), we define the *decelleration parameter*:

$$q_0 \equiv -\frac{\ddot{a}_0}{\dot{a}_0^2} a_0 \tag{4.4}$$

For matter domination, we can find this to be $q_0 = \Omega_0/2$.

4.4 Age of the Universe t_0

We can do this in a few different ways.

A rough guess is to take $1/H_0$. This gives $t_0 \approx 10^{10}$yrs.

We can use radionucleides in stars. Measure the abundances of heavy elements in stars, by looking at spectral lines. We then must add on some time for the star to have evolved to such a stage as to be able to produce such element. This gives $t_0 \approx 13.5 \pm 3 \times 10^9$yrs.

We can also use stellar evolution theory. Look at the oldest, coldest stars black body spectrum. Then, we can find out their age.

All these methods combined give:

$$t_0 \approx 13 \times 10^9 yrs \tag{4.5}$$

4.5 Ω_m

We can measure Ω_b (baryons) by counting stars, and summing over light emissions. This gives:

$$\frac{\rho_{stars}}{\rho_{crit}} \equiv \Omega_b \approx 0.04$$

We must be careful: there is a certain amount of luminous and non-luminous (baryonic) matter; both of which are included above.

4.5.1 Dark Matter

There is substantial evidence for this:

- Galaxy rotation curves: If there is only mass enclosed (i.e. none outside), then $v \propto r^{-1/2}$. However, 'outside' a galaxy, we observe $v \propto r$. Hence, there is mass outside the 'main' galaxy. This does not interact with EMR, hence we cannot 'see' it. But it does gravitate.
- Clusters of galaxies: We can measure the kinetic energy of a cluster, from their Doppler shifts. We estimate the potential energy by summing over starlight. We predict some $\langle KE \rangle = \frac{1}{2}\langle PE \rangle$ relation (the virial theorem). However, we need a lot more mass to make this work: more mass than starlight.
- Gravitational lensing: We need more mass doing the lensing, than we can account for.

All of this leads to:

$$\Omega_{DM} = 0.23 \tag{4.6}$$

Properties of DM: it must gravitate, but not interact with EMR.

Candidates: neutrinos, WIMPS, none of the above!

Thus:

$$\begin{aligned} \Omega_m &= \Omega_{luminous,b} + \Omega_{non-luminous,b} + \Omega_b & (4.7) \\ &= < 0.01 + 0.04 + 0.23 & (4.8) \\ &= 0.27 & (4.9) \end{aligned}$$

Olbers Paradox: resolved by noting that the universe has a finite age.

5 CMBR

The current temperature of the CMBR:

$$T_{CMBR,0} = 2.725 \pm 0.001K \tag{5.1}$$

The corresponding black body spectrum is uniform to 1 in 10^5, after effects of the earths motion & galaxy effects are removed.

The energy density of the CMBR is about 10x that due to starlight.

The energy per unit volume is given by:

$$\epsilon = \alpha T^4 \qquad \alpha \equiv 7.6 \times 10^{-16} Jm^{-3}K^{-4} \tag{5.2}$$

Then, the equivalent mass density of the CMBR, calculated from $\epsilon = \rho c^2$, is:

$$\rho_{CMBR,0} = 4.7 \times 10^{-31} kgm^{-3} \tag{5.3}$$

5.1 Energy Distribution

The black body spectrum is the usual, found from BE statistics.

The maximum energy photons within the BB spectrum, at that temperature, is at about $3k_BT$. However, the occupation number is highest for the lowest energies.

$\nu \propto \frac{1}{a}$. So, if, at some point in the past, when the universe was half its present size (say), then the frequency of the peak is double what it is now.