Non-Linear Physics - A Summary

1 Stability Analysis

Attractor: motion is towards, and is thus stable.
Repeller: motion is away from, and is thus unstable.

For a limit cycle to exist, the system must have at least 2 dimensions; chaos does not exist for systems whose dimension is less than 3.

1.1 Fixed Points

If we have the 1D system $\dot{x} = f(x)$, then the point at which $\dot{x} = 0$ is a fixed point:

$$f(x^*) = 0$$

Defines a fixed point. In n-D, a fixed point is such that:

$$f_i(x_1^*, x_2^*, \ldots, x_n^*) = 0 \qquad \forall\ i = 1, \ldots, n$$

Fixed points must be solved for/found algebraically.

1.2 Classification of Fixed Points

We perturb the system by a small amount, around a fixed point, and see how the system evolves: if the 'small amount' grows in time, then the fixed point is unstable; if the 'small amount' shrinks in time, then the fixed point is stable.
So, we have the construction $x(t) = x^* + \hat{x}(t)$. From a Taylor expansion, we get:

$$\frac{d\hat{x}}{dt} = \hat{x} f'(x^*)$$

Which can be solved via standard integration methods. We find the two cases:

- If $f'(x^*) < 0$, then x^* is a stable fixed point;
- If $f'(x^*) > 0$, then x^* is an unstable fixed point;

If we have a system in n-D, it becomes harder to determine the stability of the fixed point (FP). However, to do so, the theory behind it is the same:
We perturb around each fixed point by a small amount, and see how the 'small amount' changes in time: grows or shrinks for unstable or stable.

We end up having the Jacobian in an eigenvalue-type equation. We must solve the system:

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2, \\ \vdots \\ \hat{x}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \cdots \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2, \\ \vdots \\ \hat{x}_n \end{pmatrix}$$

$$= J \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2, \\ \vdots \\ \hat{x}_n \end{pmatrix} \qquad J_{ij} = \frac{\partial f_i}{\partial x_j}$$

The Jacobian entries are all evaluated at the fixed point. So, we must solve the eigenvalue equation:

$$J\hat{v} = \lambda\hat{v}$$

Which we can solve to get n-eigenvalues with eigenvectors for a system in n-dimensions.
In 3D this looks like:

$$\begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \end{pmatrix} = \lambda^i \begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \end{pmatrix}$$

Where we will get 3 eigenvalues/vectors. These are found in the standard method of finding the determinant.
If the system is in 2D, we get solutions:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} v_x^1 \\ v_y^1 \end{pmatrix} e^{\lambda^1 t} + \begin{pmatrix} v_x^2 \\ v_y^2 \end{pmatrix} e^{\lambda^2 t}$$

Thus, two eigenvalues λ, with their corresponding eigenvectors.
Remember, this procedure must be done to classify each fixed point in turn.

The actual classification of fixed points is done via the eigenvalues of the associated Jacobian. In 2D it it easy to visualise the following, but virtually impossible for higher dimensions.

- Both eigenvalues are real: $\lambda_1, \lambda_2 \in \mathbb{R}$

  If $\lambda_1, \lambda_2 > 0$, then $\hat{x}$ diverges exponentially quickly, and is hence unstable;

  If $\lambda_1, \lambda_2 < 0$, then $\hat{x}$ converges exponentially quickly, and is hence stable;

  If $\lambda_1 < 0$ and $\lambda_2 > 0$, then we have a saddle point, which is unstable.

- Both eigenvalues are complex conjugates $\lambda_1, \lambda_2 \in \mathbb{C}$.
  Infact, we have that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

  If $\alpha > 0$, then unstable spiral;

  If $\alpha < 0$, then stable spiral;

  If $\alpha = 0$, then concentric circles, which never decay.

Notice, classification of this type dosent make much sense in higher dimensions. However, its worth noting that if there is a complex eigenvalue, then its conjugate will always be present as another.

1.3 Gradient Systems

A gradient system is such that:

$$f_i(x_1, x_2, \ldots, x_n) = -\frac{\partial V}{\partial x_i}$$

Where $V(x_1, x_2, \ldots, x_n)$ is some continuous potential function.
For example, in a 2D system, we will have the equations:

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

And we see that if some $V(x, y)$ is to exist, then we must be able to satisfy:

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$$

That is exactly the same as making sure the following holds:

$$\frac{\partial \dot{x}}{\partial y} = \frac{\partial \dot{y}}{\partial x}$$

If a system is 'gradient', then there does not exist a limit cycle for the system.

1.4 Limit Cycles

A limit cycle, sometimes referred to as a closed orbit, is defined as a single, unique isolated trajectory in phase space; and dosent depend on the energy of the system.

Limit cycles may be:

- Stable: Nearby trajectories will fall towards the cycle;
- Unstable: Nearby trajectories will repel from the cycle;
- Half-stable: Trajectories on either side of the cycle do opposite things.

Limit cycles are very hard to analytically find and classify; and this is mainly done numerically. However, there are methods for ruling out the existence of limit cycles and for proving that some limit cycle exists without having to find it.

- If there exists a potential for the system (i.e. it is a gradient system) then no limit cycles exist;
- If there exists a Liapunov function $\phi(x, y)$ for the system, then no limit cycle exists;
- Use the Poincare-Bendixson theorem to prove that a limit cycle does exist.

1.4.1 Liapunov Functions

The existence of a Liapunov function $\phi(x, y)$ means that a system does not posses a limit cycle. The function must have the following properties:

- $\phi(x^*, y^*) = 0$;
- $\phi(x, y) > 0 \quad \forall \quad x, y, \neq x^*, y^*$;
- $\frac{d\phi}{dt} < 0 \quad \forall \quad x, y, \neq x^*, y^*$

However, the form of ϕ must be guessed and subsequently checked. So this process can be a little tedious!

1.4.2 The Poincare-Bendixson Theorem

This theorem states (and we have not proved it) that if R is some region in phase space, such that R does not contain any fixed points, then if some trajectory C is inside R, it is either a limit cycle, or tends towards the limit cycle after a long time.

This theorem only holds in 2D.

We construct some trapping region, where we cut out any fixed points of the system, so that: $\dot{r} > 0$ on some $r = r_1$, and $\dot{r} < 0$ on some $r = r_2$, then trajectories wil be trapped in the region $r_1 \leq r \leq r_2$.

2 Chaotic Dynamics

Chaos is defined as aperiodic behaviour in a deterministic system that exhibits sensitive dependance on initial conditions.

Chaos is not possible in a system of less than 3 dimensions. For 2D systems, then the only possible trajectories are towards either a fixed point or limit cycle or out to infinity.

2.1 Liapunov Exponent

If two trajectories start out at a small distance $\delta(0)$ apart, we want to know what $\delta(t)$ is. Numerically, it is found to be:

$$|\delta(t)|^2 = |\delta(0)|^2 e^{Lt}$$

Where L is known as the Liapunov exponent. The system has the same number of exponents as dimensions; however, it is only the one with the largest magnitude which needs to be taken into account.

We have that $L < 0$ for stability, and $L > 0$ for chaos.

We can put some limit on the divergence of the trajectories, a, say; after which trajectories have diverged 'too much'. So $|\delta(t)| \geq a$. Thus we can find the time t after which this is the case:

$$t = \frac{1}{L} \ln \left| \frac{a}{\delta(0)} \right|$$

3 Iteration Maps

A 1D map has the form:

$$f(x_n) = x_{n+1}$$

The 1D example we consider is the logistic map:

$$x_{n+1} = \lambda x_n(1 - x_n)$$

A 2D map has the form:

$$\begin{aligned} f(x_n, y_n) &= x_{n+1} \\ g(x_n, y_n) &= y_{n+1} \end{aligned}$$

The 2D example we consider is the Henon map:

$$\begin{aligned} f(x, y) &= 1 - ax^2 + y \\ g(x, y) &= bx \end{aligned}$$

3.1 Fixed Points

A fixed point is a number that we put into the function, and get it back out again:

$$f(x_n^*) = x_n^*$$

The fixed points need to be found algebraically.
A fixed point in 1D is stable if:

$$\left. \frac{df}{dx} \right|_{x=x^*} < 1$$

And unstable if greater than 1. For the logistic map, we find that fixed points are only stable on particular range of the control parameter λ.

In 2D, a fixed point is from:

$$\begin{aligned} f(x^*, y^*) = x^* \\ g(x^*, y^*) = y^* \end{aligned}$$

And can be analysed by considering the eigen-type equation:

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{\boldsymbol{x}=\boldsymbol{x}^*} \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix}$$

Where a fixed point $\boldsymbol{x}^*$ is stable if all the eigenvalues μ of the Jacobian satisfy:

$$|\mu| < 1$$

Notice how this will generalise to a system in n-dimensions. However, notation becomes clumsy.

We find that there is always a period of stability for fixed points, and stability will end when a 2-cycle is born.

3.2 2-Cycles

In 1D, a 2-cycle is such that:

$$\begin{aligned} p &= f(q) \\ q &= f(p) \\ \Rightarrow f(f(x)) &= x \end{aligned}$$

That is, we have that the system jumps between two points p, q. The 2-cycle is stable on:

$$|f'(p)f'(q)| < 1$$

Again, this gives a range of stability for the 2-cycle, and actually starts to stabilise where the fixed points become unstable. It can be shown that the 2-cycle becomes unstable as a 4-cycle starts, and that becomes unstable as an 8-cycle starts. This is known as period doubling.

For a 2D system, we have a 2-cycle at:

$$\begin{aligned} x_{n+2} &= f(x_{n+1}, y_{n+1}) = f(f(x_n, y_n), g(x_n, y_n)) \\ y_{n+2} &= g(x_{n+1}, y_{n+1}) = g(f(x_n, y_n), g(x_n, y_n)) \end{aligned}$$

And we end up with analysis of the stability of the 2-cycle being determined by:

$$\begin{pmatrix} \hat{x}_{n+2} \\ \hat{y}_{n+2} \end{pmatrix} = J^{(2)} \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix}$$

Where:

$$J^{(2)} \equiv \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{\boldsymbol{x}=\boldsymbol{p}} \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{\boldsymbol{x}=\boldsymbol{q}}$$

The product of the Jacobians, each evaluated at each position in the 2-cycle.
We have that stability is determined by all eigenvalues μ of the Jacobian $J^{(2)}$ satisfying:

$$|\mu| < 1$$

One can see how this generalises to an n-cycle, for a system in N-dimensions; although the notation becomes tricky!

In both 1D, 2D cases, as a period 2^ℓ object becomes unstable, another or period $2^{\ell+1}$ is born.

3.3 Chaos & Fiegenbaum

In the logistic map, the parameter which determines stability is λ. Let Λ_ℓ be the value of λ for which a period-2^ℓ object is born. Thus, we find a load of $\Lambda_\ell' s$. Fiegenbaum wrote down:

$$\delta_n = \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}}$$

And found that as $n \to \infty$, $\delta_n \to \delta$. We denote this constant δ the Fiegenbaum number:

$$\delta \equiv \lim_{n\to\infty} \delta_n = 4.6692016\ldots$$

This is a constant for a massive amount of maps. If $n = \infty$, then Λ_∞ is the value of λ for which we get a period-2^∞ object. Which is just chaos. We can numerically find Λ_∞. For $\Lambda > \Lambda_\infty$, we find systems where periods start at prime numbers, the period double to infinity $3^\infty, 7^\infty, 13^\infty \ldots$.
We can find the Liapunov exponent for the system L via:

$$L = \frac{1}{n}\sum_{i=0}^{n-1} \ln|f'(x_i)|$$

And find when $L > 0$, with respect to λ. Hence, we find when the chaotic periods start and end.

4 Fractals

Names of fractals we have discussed: von Koch curve/snoflake, Cantor set, Sierpinski Gasket.

4.1 Similarity Dimension

Suppose in producing a fractal, or any pattern actually, we do operations on a shape, where each iteration produces m copies, each being a factor of r smaller than the original. Then, the 'similarity dimension' is:

$$d = \frac{\ln m}{\ln r}$$

A fractal is a shape for which d is non-integer.

5 Further Aspects

We are able to show that a volume V in an n-dimensional phase space evolves in time as:

$$\frac{dV}{dt} = \int_V \sum_i \frac{\partial f_i}{\partial x_i}\, dV$$

In 3D, thus is just the integral of the divergence of the vector whose elements are the functions of the system.
We find that volumes, for non-conservative systems, shrink to zero. Thus showing that non-conservative systems always evolve to some limiting set; of fixed points, limit cycles or strange attractors.