Non-Linear Dynamics

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Contents

1 General Features of Dynamical Systems

1.1 The Unforced, Overdamped Pendulum

We have a standard pendulum, of length l, with a bob mass m. We have thus have equations of motion:

$$\begin{aligned} F = ma &= ml\ddot{\theta} \\ &= -mg\sin\theta - bl\dot{\theta} + f(t) \end{aligned}$$

Where b is some frictional force, and $f(t)$ some forcing term (which we will be neglecting here). Hence, $bl\dot{\theta}$ is the damping term: frictional force multiplied by the velocity.
Hence, re-writing:

$$\ddot{\theta} + \left(\frac{b}{m}\right)\dot{\theta} + \left(\frac{g}{l}\right)\sin\theta = \frac{1}{ml}f(t)$$

From now on, we use unforced: $f(t) = 0$.
Overdamping means that the frictional force is so strong that we ignore the $\ddot{\theta}$ term: $\frac{b}{m}\dot{\theta} >> \ddot{\theta}$.
Hence, we have:

$$\left(\frac{b}{m}\right)\dot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0 \qquad (1.1)$$

$$\Rightarrow \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0 \qquad (1.2)$$

Now, we introduce a 'dimensionless time' parameter τ, to tidy things up:

$$\begin{aligned} \tau &\equiv \frac{t}{T} \\ T &\equiv \frac{bl}{mg} \end{aligned}$$

Hence, using the chain rule, we have:

$$\frac{d\theta}{dt} = \frac{d\tau}{dt}\frac{d\theta}{d\tau} \qquad (1.3)$$

$$= \frac{1}{T}\frac{d\theta}{d\tau} \qquad (1.4)$$

$$= \frac{mg}{bl}\frac{d\theta}{d\tau} \qquad (1.5)$$

Hence, our equation of motion (1.2) becomes:

$$\frac{d\theta}{d\tau} + \sin\theta = 0 \qquad (1.6)$$

Now, this looks horrible, and can be integrated to give $\tau = \ln|\csc\theta + \cot\theta| + c$. However, this analytical solution dosent show us what we are interested in: long-term behaviour.
To do this analysis, there are a number of ways to do it:
First, is to look at a plot of $\dot{\theta}$ against θ (this is a phase-plane plot). On the graph, we have

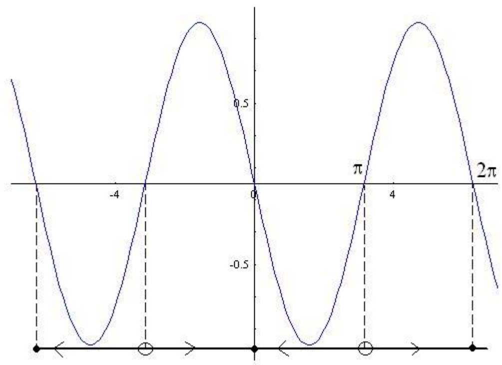


Figure 1: A plot of $\dot{\theta}$ against θ. Notice the stationary points: where $\dot{\theta} = 0$.

dropped lines down where $\dot{\theta} = 0$, i.e. where there is no motion. The arrows indicate the direction of increasing θ with time.
Points for which $\dot{\theta} = 0$ are defined as 'fixed points'.
There are 2 types of fixed point here:
Attractors, or 'sinks': motion is towards them. These are indicated by black dots, and occour at $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$
Repellors, or 'sources': motion is away from them. These are indicated by white dots, and are at $\theta = \pm\pi, \pm 3\pi, \ldots$
Note, the repellors are actually fixed points: i.e. no motion occurs at the repellors. If motion starts of exactly at a repellor, it will stay there; If motion starts near a repellor, then motion will be repelled from the fixed point, thus we see that a repellor is unstable. If motion start near an attractor, then trajectories will tend toward the attractor; hence, an attractor is a stable fixed point. Unless initially at a repellor, the final state of the system will be at an attractor.
Suppose we start at $\theta_0 = -\frac{2\pi}{3}$, we have that $\dot{\theta} > 0$, and motion will increase till $\theta = -\frac{\pi}{2}$, then decrease to zero at $\theta = 0$; the position of the nearest attractor.

Another way to visualise the dynamics is to work in terms of potential energy.
The unforced equation has the form:

$$ml\ddot{\theta} = -\frac{1}{l}\frac{dV}{d\theta} - bl\dot{\theta} \tag{1.7}$$

Where V is the potential energy: $V = -mgl\cos\theta + mgl$, measured from the lowest point. Just to check that this is correct, we see if we can get back to the original form of the equation. So:

$$\begin{aligned} \frac{dV}{d\theta} &= mgl\sin\theta & (1.8) \\ \Rightarrow ml\ddot{\theta} &= -mgl\sin\theta - bl\dot{\theta} & (1.9) \end{aligned}$$

Which is what we had before; so check completed.
Now, going to the overdamped case, we have:

$$\frac{1}{l}\frac{dV}{d\theta} + bl\dot{\theta} = 0 \tag{1.10}$$

We now introduce a dimensionless potential:

$$u(\theta) = \frac{V(\theta)}{mgl} = 1 - \cos\theta \tag{1.11}$$

Now, from before, if we use the time dimensionless form of the overdamped equation of motion (1.6):

$$\frac{d\theta}{d\tau} + \sin\theta = 0 \tag{1.12}$$

Thus, using our dimensionless potential (1.11), this becomes:

$$\frac{d\theta}{d\tau} = -\frac{du}{d\theta} \tag{1.13}$$

Now, plotting a graph of $u(\theta) = 1 - \cos\theta$ against θ will show how the potential energy varies throughout the motion, and will hence provide a way to see if the current state of the system is stable or unstable.

Notice that again, in the potential plot, we have attractors & repellors: attractors are minima and repellors maxima of the potential.
If a potential exists (as it isnt always possible to do this), we can think of this as a ball moving in a potential. However, as the damping term is so large, there is no oscillation around equilibrium points.
In overdamped problems, there is no overshooting or damped-oscillations.

$$\frac{du}{d\tau} = \frac{du}{d\theta}\frac{d\theta}{d\tau} \tag{1.14}$$

$$= -\left(\frac{du}{d\theta}\right)^2 \leq 0 \tag{1.15}$$

So u always decreases with time, unless $\frac{du}{d\theta} = 0$, at which point we have a 'fixed point'.

2 Linear Stability Analysis

To find the nature of fixed points, first, we find the fixed points, which is done algebraically.
In the case before, where the equation of motion was $\frac{d\theta}{d\tau} = -\sin\theta$, we had fixed points where $\frac{d\theta}{d\tau} = 0$.
Hence, we have fixed points where $-\sin\theta = 0$. Or, where $\theta = n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$.
We call a fixed point θ^*. We write:

$$\theta(\tau) = \theta^* + \hat{\theta}(\tau) \tag{2.1}$$

Where $\hat{\theta}$ is some small deviation (perturbation) from the fixed point θ^*. Here, $\theta^* = n\pi$.
Hence, we can see that:

$$\frac{d\theta}{d\tau} = \frac{d\hat{\theta}}{d\tau} = -\sin\theta \tag{2.2}$$

If we do a Taylor expansion of $\sin\theta$, we have:

$$\begin{aligned}\sin\theta &= \sin(\theta^* + \hat{\theta}) &(2.3)\\ &= \sin\theta^* + \hat{\theta}\cos\theta^* + \ldots &(2.4)\end{aligned}$$

Hence, we have that:

$$\begin{aligned}\frac{d\hat{\theta}}{d\tau} &= -\sin\theta &(2.5)\\ &= -(\sin\theta^* + \hat{\theta}\cos\theta^*) &(2.6)\end{aligned}$$

But, by definition $\sin\theta^* = 0$ at a fixed point. Hence,

$$\begin{aligned}\frac{d\hat{\theta}}{d\tau} &= -\hat{\theta}(\cos\theta^*) &(2.7)\\ &= -\hat{\theta}(\cos n\pi) &(2.8)\\ &= -\hat{\theta}(-1)^n &(2.9)\end{aligned}$$

Now, suppose that n is odd. Then we have:

$$\begin{aligned}\frac{d\hat{\theta}}{d\tau} &= \hat{\theta} &(2.10)\\ \Rightarrow \hat{\theta}(\tau) &= Ae^{\tau} = \theta(0)e^{\tau} &(2.11)\end{aligned}$$

Thus, if we start very near the fixed point (with n odd) at $\tau = 0$, we move away from it exponentially fast. Hence, this fixed point is unstable.
Suppose n is even. Similarly we have:

$$\begin{aligned}\frac{d\hat{\theta}}{d\tau} &= -\hat{\theta} &(2.12)\\ \Rightarrow \hat{\theta}(\tau) &= Ae^{-\tau} = \theta(0)e^{-\tau} &(2.13)\end{aligned}$$

Thus, for this fixed point, if we start slightly away from it, we will move back to it exponentially, as time progresses. Hence, it is a stable fixed point.

So, what we have derived is how some small perturbation $\hat{\theta}$ evolves in time, if we start very close to some fixed point. We are thus assessing the stability of a fixed point.

2.1 General Linear Stability Analysis in 1D

Lets generalise in 1D.
Suppose we have that:

$$\dot{x} = f(t) \tag{2.14}$$

Which is known as a first order system.
Fixed points are where $\dot{x} = \frac{dx}{dt} = 0$, are solutions of $f(x^*) = 0$. Lets look at one fixed point x^*:

$$x(t) = x^* + \hat{x}(t) \tag{2.15}$$

Where $\hat{x}$ is some small movement around the fixed point x^*. Hence, we have that:

$$\frac{dx}{dt} = \frac{d\hat{x}}{dt} \tag{2.16}$$

And if we Taylor expand $f(x)$ around x^*, we get:

$$\begin{aligned} f(x) &= f(x^*) + \hat{x}\frac{df}{dx}_{x=x^*} + \frac{\hat{x}^2}{2!}\frac{d^2 f}{dx^2}_{x=x^*} \\ &= \hat{x}\frac{df}{dx}_{x=x^*} \\ \Rightarrow \frac{d\hat{x}}{dt} &= \hat{x}f'(x^*) \equiv A\hat{x} \\ \Rightarrow \frac{d\hat{x}}{dt} &= A\hat{x} \end{aligned}$$

Remebering that $f(x^*) = 0$ is the definition of a fixed point.
To start the analysis of the fixed points, we look at a couple of cases:
If $f'(x^*) > 0$ (i.e. $A > 0$), then we will move away from a fixed point:

$$\hat{x}(t) = \hat{x}(0)e^{At} \tag{2.17}$$

And is hence an unstable fixed point.
Conversely, if $f'(x^*) < 0$ (i.e. $A < 0$), then we will move towards a fixed point:

$$\hat{x}(t) = \hat{x}(0)e^{-At} \tag{2.18}$$

And is hence a stable fixed point.

Example Let $\dot{x} = x - x^3$. Then, we have fixed points where $\dot{x} = 0$; i.e. $x^* - (x^*)^3 = 0$. Hence, we have fixed points at $x^* = 0, -1, +1$.
Thus, we have that $f'(x) = 1 - 3x^2$, and $f'(x^* = 0) = 1 > 0$, and is hence unstable. We also have that $f'(x^* = \pm 1) = -2 < 0$, and is hence stable.

2.2 Fixed Points and Their Stability, in General

2.2.1 Reducing Differential Equations to First Order

We have an example of a Newtonian system: the equations of motion for N particles are:

$$m_i \frac{d^2 r_i}{dt^2} = F_i(r_1, r_2, \ldots, r_N) \qquad i = 1, 2, \ldots, N$$

Now, if we define a momentum p_i, we can reduce this second order equation into two coupled first orders:

$$\begin{aligned} m_i \frac{dr_r}{dt} &= p_i \\ \frac{dp_i}{dt} &= F_i(r_1, p_1; r_2, p_2; \ldots; r_N, p_N) \end{aligned}$$

Hence, we say that we can write a system in terms of a general set of equations:

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, \dots, x_N) \\ &\vdots \\ \frac{dx_N}{dt} &= f_N(x_1, x_2, \dots, _N)\end{aligned}$$

We can look at a couple of examples of reducing systems to first order differential equations:

Pendulum We look at the pendulum again. Its full (unforced) equation of motion is:

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{l}\sin\theta = 0$$

Now, to reduce this second-order equation to single-orders, we introduce Ω, as well as 2 dimensionless variables τ, γ:

$$\begin{aligned}\tau &\equiv \frac{t}{\sqrt{\frac{l}{g}}} \\ \gamma &\equiv \frac{b}{m}\sqrt{\frac{l}{g}} \\ \Omega &\equiv \frac{d\theta}{d\tau}\end{aligned}$$

Hence, under these equations, the equation of motion can be written as two first-order equations:

$$\begin{aligned}\frac{d\theta}{dt} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\gamma\Omega - \sin\theta\end{aligned}$$

Populations If N_1, N_2 are the numbers of individuals of 2 different speices in competition with each other, we can write some system of equations linking the populations:

$$\begin{aligned}\frac{dN_1}{dt} &= N_1(r_1 - a_{11}N_1 - a_{12}N_2) \\ \frac{dN_2}{dt} &= N_2(r_2 - a_{22}N_2 - a_{21}N_1)\end{aligned}$$

Now, we consider classes of such systems: Gradient and Hamiltonian systems.

2.2.2 Gradient Systems

These are such that:

$$f_i(x_1, \dots, x_n) = -\frac{\partial V}{\partial x_i} \tag{2.19}$$

We can think of such systems using a mechanical analogy: particle in overdamped n-D space, moving in a potential V. We can consider a couple of examples:

Overdamped Pendulum We can write in a potential $U(\theta)$, into the equations of motion:

$$\begin{aligned}\frac{d\theta}{d\tau} &= -\frac{dU}{d\theta} \\ U(\theta) &= 1-\cos\theta\end{aligned}$$

This has been done in a previous example. The existence of a potential means that the overdamped pendulum is a 'gradient system'.

Damped Pendulum Here, we have equations of motion:

$$\begin{aligned}\frac{d\theta}{d\tau} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\gamma\Omega-\sin\theta\end{aligned}$$

Hence, as two equations, we have a 2D problem. In dimensions above one, it isnt always possible to find potentials. Lets see if we can find a V.
Now, if a $V(\theta,\Omega)$ exists, it must satisfy both equations:

$$\begin{aligned}\frac{dV}{d\theta} &= -\Omega \\ \frac{dV}{d\Omega} &= \gamma\Omega+\sin\theta\end{aligned}$$

Now, there are two ways in which we could check for the existence of such a V.
First, is to differentiate each equation, with respect to the other. Noting that for a continuous function $f(x_i)$, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$. This is what we check for:

$$\begin{aligned}\frac{\partial^2 V}{\partial\Omega\partial\theta} &= -1 \\ \frac{\partial^2 V}{\partial\theta\partial\Omega} &= \cos\theta \\ \Rightarrow \frac{\partial^2 V}{\partial\Omega\partial\theta} &\neq \frac{\partial^2 V}{\partial\theta\partial\Omega}\end{aligned}$$

Hence, we find that it is not a gradient system; as no potential exists.
The other way of checking for a potential V is to integrate the system directly.

2.2.3 Hamiltonian Systems

These are non-dissipative/conservative classical mechanical systems. They only exist if the number of variables n is even.
Here, the coordinates $(x_1,\ldots,x_{2m})$ are split into $(q_1,\ldots,q_m)$ and $(p_1,\ldots,p_m)$: generalised coordinates and momenta. They are linked via the Hamiltonian H:

$$\frac{\partial p_\alpha}{\partial t} = -\frac{\partial H}{\partial q_\alpha} \quad (2.20)$$

$$\frac{\partial q_\alpha}{\partial t} = \frac{\partial H}{\partial p_\alpha} \quad (2.21)$$

For example, a particle moving in a 3D potential $V(x, y, z)$ has Hamiltonian

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$
$$\dot{p_\alpha} = -\frac{\partial V}{\partial q_\alpha} \quad \dot{q_\alpha} = \frac{p_\alpha}{m}$$

Notice that these give the Newtonian result:

$$\dot{p_\alpha} = \ddot{q_\alpha} m = -\frac{\partial V}{\partial q_\alpha}$$

We do not consider Hamiltonian systems in any subsequent discussions; and they do not form any part of this course. This is just here to give an example.

2.3 Analysis of the Undamped Pendulum: Phase Space

Here, we have the two equations which we have previously derived:

$$\begin{aligned} \frac{d\theta}{d\tau} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\sin\theta \end{aligned}$$

We analyse this system via drawing a 'phase portrait'.
We can see that for small θ, we have SHM around the points $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$ these are the positions of the attractor fixed-points found previously. On phase space, these are small circles, centred on the fixed attractors. These motions have small energy.
If we give the pendulum a ruddy great whack the pendulum will swing around loads, and wont stop: they dont pass through a fixed point - the trajectories dont cross fixed points. The pendulum will now have a lot of energy.
The energy is $E = \frac{1}{2}\Omega^2 + u(\theta)$, where $u(\theta) = 1 - \cos\theta$. When $\theta = \pi$, we have $E = E_0 = 2$: this corresponds to where the mass just about makes it to $\theta = \pi$. Hence, we can write an equation for $\Omega(\theta)$, to find the equations of the trajectories through the fixed points at $\theta = \pm\pi, \pm 3\pi, \ldots$: we can write:

$$\begin{aligned} 2 &= \frac{1}{2}\Omega^2 + (1 - \cos\theta) \\ \Rightarrow \Omega &= \pm\sqrt{2(1 + \cos\theta)} \\ &= \pm 2\cos\frac{\theta}{2} \end{aligned}$$

In this analysis: the curves are known as 'trajectories', in a phase diagram. The Cartesian coordinates Ω, θ define the phase plane (or space, generally).
If enough initial conditions are given (as many dimensions as the phase space), then the solution to the differential equations is unique (which implies that classical mechanics is entirely predictable: in principle, but not nessecarily in practice).
Trajectories in phase space cannot cross. If they did, then future behaviour of the system would not be unique. Hence, the lines drawn in the $E = E_0$ case cannot be real trajectories - lines like

this are called a 'separatrix'.
If we have a damping term, then trajectories will spiral in to an attractor. $\frac{d\Omega}{d\theta}$ gives the slopes of the trajectory:

$$\begin{aligned}\frac{d\Omega}{d\theta} &= \frac{\frac{d\Omega}{d\tau}}{\frac{d\theta}{d\tau}} \\ &= \frac{-\gamma\theta - \sin\theta}{\Omega} \\ &= -\gamma - \frac{\sin\theta}{\Omega}\end{aligned}$$

More generally, we can find trajectories from direction field ($\frac{d\Omega}{d\theta}$ in this case).
All this applies to the general case:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n)\end{aligned}$$

Where we have an n-dimensional phase space. We will hence have $(n-1)$ direction fields:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, \ldots, x_n)}{f_1(x_1, \ldots, x_n)}$$

If the system of equations are a function of time (e.g. driving pendulum), then the system is called *non-autonomous*. Conversely, if the system is not a function of time, the system is called *autonomous*.
In a non-autonomous system, phase space no longer consists of distinct non-intersecting curves. To get around this, we introduce another dimension: $x_{n+1} = t$. Hence, we have:

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n, x_{n+1}) \\ \frac{dx_{n+1}}{dt} &= f_{n+1}(x_1, \ldots, x_n, x_{n+1}) = 1\end{aligned}$$

Now, an example of this, is the damped, driven pendulum:

$$\frac{d^2\theta}{d\tau^2} + \gamma\frac{d\theta}{d\tau} + \sin\theta = G\cos(\omega_D\tau)$$

Where we call the parameters G (driving amplitude), ω_D (driving frequecy) and γ (damping coefficient) **control parameters**. Now, if we do our usual thing of writing two second order equations:

$$\frac{d\theta}{d\tau} = \Omega \tag{2.22}$$

$$\frac{d\Omega}{d\tau} = -\gamma\Omega - \sin\theta + G\cos(\omega_D\tau) \tag{2.23}$$

We have a problem, in that (2.23) is non-autonomous: it is function of time (τ). So, we introduce a third equation, making the system 3-dimensional:

$$\frac{d\theta}{d\tau} = \Omega \quad (2.24)$$
$$\frac{d\Omega}{d\tau} = -\gamma\Omega - \sin\theta + G\cos(\phi) \quad (2.25)$$
$$\frac{d\phi}{d\tau} = \omega_D \quad (2.26)$$

Where we see that $\phi \equiv \omega_d \tau$. Hence, we have constructed an autonomous system from a non-autonomous system, by introducing an extra dimension.

2.4 Fixed Points

We start by looking at the example of the system:

$$\dot{x} = \alpha x - \gamma y^2 - \delta x^3$$
$$\dot{y} = \beta y - \gamma y x^2 - \epsilon y^3$$

Where the constants are all > 0. We can easily show that this is a gradient system, by showing that, if $\dot{x} = -\frac{\partial V}{\partial x}$, and $\dot{y} = -\frac{\partial V}{\partial y}$:

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y}$$
$$\Rightarrow \frac{\partial \dot{x}}{\partial y} = \frac{\partial \dot{y}}{\partial x}$$

Which can easily be shown.
So now, we look for the fixed point, i.e. where $\dot{x} = \dot{y} = 0$.
The trivial one is where $x^* = y^* = 0$.
The next set is for $x^* = 0$. Hence, this leaves us with $\beta = \gamma(x^*)^2 + \epsilon(y^*)^2 \Rightarrow y^* = \pm\sqrt{\frac{\beta}{\epsilon}}$.
The next set is when $y^* = 0 \Rightarrow x^* = \pm\sqrt{\frac{\alpha}{\delta}}$.
The final one is for both $x^*, y^* \neq 0$. This is below, at the end of the list.
Hence, all the fixed points are at:

$$(0,0)$$
$$\left(0, \pm\sqrt{\frac{\beta}{\epsilon}}\right)$$
$$\left(\pm\sqrt{\frac{\alpha}{\delta}}, 0\right)$$
$$\left(\pm\sqrt{\frac{\beta\gamma - \alpha\epsilon}{\gamma^2 - \delta\epsilon}}, \pm\sqrt{\frac{\alpha\epsilon - \beta\gamma}{\gamma^2 - \delta\epsilon}}\right)$$

And, by plotting & analysing the potential function $V(x,y)$ we would be able to see which are stable/unstable.

If we go back to the general set of equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n)\end{aligned}$$

If we look at linear stability analysis: we perturb each fixed point x_i^* by a small amount $\hat{x}_i$:

$$\begin{aligned}x_1 &= x_1^* + \hat{x}_1 \\ &\vdots \\ x_n &= x_n^* + \hat{x}_n\end{aligned}$$

We can write the general Taylor expansion about each fixed point:

$$\frac{d\hat{x}_i}{dt} = f_i(\underline{x}^*) + \frac{\partial f_i}{\partial x_1}_{\underline{x}=\underline{x}^*} \hat{x}_1 + \ldots + \frac{\partial f_i}{\partial x_n}_{\underline{x}=\underline{x}^*} \hat{x}_n + \ldots \tag{2.27}$$

Where we havnt written higher-order differentials.
Now, $f_i(\boldsymbol{x}^*) = 0$, due to the definition of a fixed point. Hence, We can write the system of equations which will be a result of (2.27), as a matrix

$$\begin{pmatrix} \frac{d\hat{x}_1}{dt} \\ \vdots \\ \frac{d\hat{x}_n}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{pmatrix}$$

Which can be compressed into suffix notation:

$$\frac{d\hat{x}_i}{dt} = \frac{\partial f_i}{\partial x_j}\hat{x}_j$$

2.5 Linear Stability Analysis of Multivariable Systems

2.5.1 2D General Case

Suppose that the system is given by:

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}$$

We first find the fixed points: which is done algebraically, by setting $\dot{x} = \dot{y} = 0$. Once the fixed points x^*, y^* have been found, the linear stability analysis will have to be done for each fixed point. Now, for each set of fixed points, introduce small perturbations $\hat{x}(t), \hat{y}(t)$:

$$\begin{aligned}x(t) &= x^* + \hat{x}(t) \\ y(t) &= y^* + \hat{y}(t)\end{aligned}$$

Now, to find out how the perturbations evolve in time, substitute them back into the original differential equations; remembering that $\dot{x}^* = \dot{y}^* = 0$, as the fixed points are time-independant. Hence, we have that:

$$
\begin{aligned}
\hat{x} &= \dot{\hat{x}} = f(x^* + \hat{x}, y^* + \hat{y}) \\
\hat{y} &= \dot{\hat{y}} = g(x^* + \hat{x}, y^* + \hat{y})
\end{aligned}
$$

We Taylor expand each to first order (only the x-equation is done fully here):

$$
\begin{aligned}
\dot{\hat{x}}(t) &= f(x^*, y^*) + \hat{x}(t)\frac{\partial f}{\partial x} + \hat{y}(t)\frac{\partial f}{\partial y} \\
\Rightarrow \dot{\hat{x}}(t) &= \hat{x}(t)\frac{\partial f}{\partial x} + \hat{y}(t)\frac{\partial f}{\partial y} \\
\dot{\hat{y}}(t) &= \hat{y}(t)\frac{\partial g}{\partial x} + \hat{y}(t)\frac{\partial g}{\partial y}
\end{aligned}
$$

Using that, by definition $f(x^*, y^*) = 0$. Hence, as each of the differentials are to be evaluated at the fixed point, we have that:

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{x}\left.\frac{\partial f}{\partial x}\right|_{FP} + \hat{y}\left.\frac{\partial f}{\partial y}\right|_{FP} \\
\dot{\hat{y}} &= \hat{x}\left.\frac{\partial g}{\partial x}\right|_{FP} + \hat{y}\left.\frac{\partial g}{\partial y}\right|_{FP}
\end{aligned}
$$

Now, this system of equations may be written in matrix form:

$$
\frac{d}{dt}\begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{FP} \begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \end{pmatrix}
$$

The 2x2 matrix is called the **Jacobian**, and is simply:

$$
J = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{FP} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}
$$

Where each element is understood to be evaluated at the fixed point. The Jacobian is sometimes referred to as a stability matrix. Hence, we have that our system of equations is:

$$
\frac{d\hat{x}}{dt} = J_{11}\hat{x}(t) + J_{12}\hat{y}(t) \tag{2.28}
$$

$$
\frac{d\hat{y}}{dt} = J_{21}\hat{x}(t) + J_{22}\hat{y}(t) \tag{2.29}
$$

The solutions to such equations involves exponentials, and we quote the solution to be:

$$
\hat{x}(t) = \hat{x}^{(1)}e^{\lambda_1 t} + \hat{x}^{(2)}e^{\lambda_2 t} \tag{2.30}
$$

$$
\hat{y}(t) = \hat{y}^{(1)}e^{\lambda_1 t} + \hat{y}^{(2)}e^{\lambda_2 t} \tag{2.31}
$$

Upon substitution of (2.30) and (2.31) into (2.28) and (2.29), we have that the constants $\hat{x}^{(i)}, \hat{y}^{(i)}$ and λ_i must satisfy:

$$
\begin{aligned}
J_{11}\hat{x}^{(i)} + J_{12}\hat{y}^{(i)} &= \lambda_i \hat{x}^{(i)} \\
J_{21}\hat{x}^{(i)} + J_{22}\hat{y}^{(i)} &= \lambda_i \hat{y}^{(i)}
\end{aligned}
$$

As long as $\lambda_i \neq \lambda_j$. This again can be expressed in terms of matrices:

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \hat{x}^{(i)} \\ \hat{y}^{(i)} \end{pmatrix} = \lambda_i \begin{pmatrix} \hat{x}^{(i)} \\ \hat{y}^{(i)} \end{pmatrix}$$

This then shows us that λ_i must be the eigenvalue of the Jacobian, corresponding to the eigenvectors $\hat{x}^{(i)}, \hat{y}^{(i)}$ (of the Jacobian).
Thus, to carry out linear stability analysis on a 2D system, we need to:

- Find the eigenvalues of the Jacobian J: λ_1, λ_2.
- Find the eigenvectors corresponding to each eigenvalue.

To determine the stability of each fixed point.
Infact, this method generalises to any number of dimensions, but we shall see that later.

Then, we have that the linear stability of a particular fixed point is given by:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x}^{(1)} \\ \hat{y}^{(1)} \end{pmatrix} e^{\lambda_1 t} + \begin{pmatrix} \hat{x}^{(2)} \\ \hat{y}^{(2)} \end{pmatrix} e^{\lambda_2 t}$$

Notice that the stability of a fixed point is entirely determined from the eigenvalues of the Jacobian. For instance, if $\Re(\lambda_1, \lambda_2) > 0$, then the exponentials blow up, and the system is unstable. If $\Re(\lambda_1, \lambda_2) < 0$, then the exponentials decay in time, hence the system is stable. Here, we take the real parts. If there are no real parts to an eigenvalue, odd things happen. If the eigenvalues have different signs, then the fixed point is a saddle point.

2.5.2 nD General Linear Stability Analysis

Suppose we have the system of equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_i &= f_i(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

And, if we define $f_i(x_1, x_2, \ldots, x_n) \equiv f_i(\mathbf{x})$. Again, the fixed points are found $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, and a small perturbation $\mathbf{x} = \mathbf{x}^* + \hat{\mathbf{x}}(t)$ applied to the fixed point. Again, doing a Taylor expansion around the fixed point, and keeping only linear perturbation terms, gives rise to the time evolution of the perturbations, which we write in matrix form:

$$\frac{d}{dt} \begin{pmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_i(t) \\ \vdots \\ \hat{x}_n(t) \end{pmatrix} = \left. \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_j} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_j} & \cdots & \frac{\partial f_i}{\partial x_n} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_j} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \right|_{FP} \begin{pmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_j(t) \\ \vdots \\ \hat{x}_n(t) \end{pmatrix} \tag{2.32}$$

Where the nxn Jacobian matrix has entries:

$$J_{ij} = \frac{\partial f_i}{\partial x_j} \tag{2.33}$$

Thus, (2.32) in suffix notation, is equivalent to:

$$\frac{d\hat{x}_i}{dt} = \left.\frac{\partial f_i}{\partial x_j}\right|_{FP} \hat{x}_j \tag{2.34}$$

$$= J_{ij}\hat{x}_j \tag{2.35}$$

And the J_{ij} are evaluated at the fixed point.
Hence, now, we have that the general solution to these equations is:

$$\begin{pmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_i(t) \\ \vdots \\ \hat{x}_n(t) \end{pmatrix} = \begin{pmatrix} \hat{x}_1^{(1)}(t) \\ \vdots \\ \hat{x}_i^{(1)}(t) \\ \vdots \\ \hat{x}_n^{(1)}(t) \end{pmatrix} e^{\lambda_1 t} + \ldots + \begin{pmatrix} \hat{x}_1^{(j)}(t) \\ \vdots \\ \hat{x}_i^{(j)}(t) \\ \vdots \\ \hat{x}_n^{(j)}(t) \end{pmatrix} e^{\lambda_j t} + \ldots + \begin{pmatrix} \hat{x}_1^{(n)}(t) \\ \vdots \\ \hat{x}_i^{(n)}(t) \\ \vdots \\ \hat{x}_n^{(n)}(t) \end{pmatrix} e^{\lambda_n t} \tag{2.36}$$

And again, in suffix notation, (2.36) is equivalent to:

$$\hat{x}_i(t) = \hat{x}_i^{(j)} e^{\lambda_j t} \tag{2.37}$$

Hence, we have that the eigenvalue λ_j corresponds to the eigenvector $\hat{x}_i^{(j)}$, of the Jacobian J. So again, if any one of the real parts of the eigenvalues is positive, the fixed point will be unstable, as the perturbation will grow in time.
To show that (2.36) is a solution to (2.32), we differentiate (2.36); If we write the eigenvector $\hat{x}_i^{(j)} = \mathbf{v}_j$, so that we have:

$$\dot{\hat{\mathbf{x}}}(t) = \frac{d}{dt}\hat{\mathbf{x}}(t) = \mathbf{v}_1\lambda_1 e^{\lambda_1 t} + \ldots + \mathbf{v}_i\lambda_i e^{\lambda_i t} + \ldots + \mathbf{v}_n\lambda_n e^{\lambda_n t} \tag{2.38}$$

And, if we multiply (2.36) by the Jacobian:

$$\mathbf{J}\hat{\mathbf{x}} = \mathbf{J}\mathbf{v}_1 e^{\lambda_1 t} + \ldots + \mathbf{J}\mathbf{v}_i e^{\lambda_i t} + \ldots + \mathbf{J}\mathbf{v}_n e^{\lambda_n t} \tag{2.39}$$

And, subtracting (2.39) - (2.38):

$$\mathbf{J}\hat{\mathbf{x}} - \dot{\hat{\mathbf{x}}}(t) = \ldots + (\mathbf{J}\mathbf{v}_i e^{\lambda_i t} - \mathbf{v}_i\lambda_i e^{\lambda_i t}) + \ldots \tag{2.40}$$

Hence, we see that if $\mathbf{v}_i$ is chosen to be an eigenvector of the Jacobian, with eigenvalue λ_i, then we have shown that our supposed solution is the solution.
To see this solution in suffix notation (we drop hats, and brackets around upper index, for clarity!):
We have two the equations:

$$\dot{x}_i = J_{ij}x_j \tag{2.41}$$

$$x_i = x_i^k e^{\lambda_k t} \tag{2.42}$$

Where we have presupposed that (2.42) is a solution to (2.41). Now, if we differentiate (2.42), with respect to time, and also multiply (2.42) by J_{ij}:

$$\begin{aligned} \dot{x}_i &= x_i^k \lambda_k e^{\lambda_k t} && (2.43) \\ J_{ij} x_i &= J_{ij} x_i^k e^{\lambda_k t} && (2.44) \end{aligned}$$

And subtract these two:

$$\begin{aligned} (\dot{x}_j - J_{ij} x_j) &= x_j^k \lambda_k e^{\lambda_k t} - J_{ij} x_j^k e^{\lambda_k t} && (2.45) \\ &= (x_j^k \lambda_k - J_{ij} x_j^k) e^{\lambda_k t} && (2.46) \end{aligned}$$

So, if we want (2.42) a solution to (2.41), we have that:

$$\begin{aligned} \dot{x}_j - J_{ij} x_j &= 0 && (2.47) \\ \Rightarrow J_{ij} x_j^k &= x_j^k \lambda_k && (2.48) \end{aligned}$$

Now, if you expand this out into a 3x3 matrix:

$$\begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{33} \\ J_{31} & J_{22} & J_{33} \end{pmatrix} \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix} = \lambda_k \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix} \tag{2.49}$$

Which is a standard eigen-equation, where the index k now indicates different eigenvalues/vectors.

2.5.3 Classification of Fixed Points

If we obtain the eigenvalues λ_1, λ_2 from the associated 2D Jacobian (i.e. a 2D system), we can classify the fixed point. There are various possibilities, all of the below assume that $\lambda_1 \neq \lambda_2$:

$\lambda_1, \lambda_2 \in \mathbb{R}$ and both negative. Here, imagine a sink; with the two eigenfunctions acting as basis vectors. The direction on the eiegenvectors corresponds to the sign on the eigenvalue (hence, as both negative, both point towards the origin). The eigenvector corresponding to the least negative eigenvalue will dominate the attraction. Thus, a 'stable node'.

$\lambda_1, \lambda_2 \in \mathbb{R}$ and both positive. Here, we have a source: so an unstable node. Both eiegnvectors are pointing away from the origin. Again, the eigenvector with the largest eigenvalue will dominate.

$\lambda_1, \lambda_2 \in \mathbb{R}$ but different sign. Here, we have a saddle point: the positive eigenvalue will give an eigenvector pointing away from the origin; and the negative eigenvalue towards the origin. Notice: if we start near the negative eigenvalue (but not on it), motion will be deflected away along the positive eigenvalue eventually - i.e. motion to begin with may move towards the node, but will move away from it after a while. Motion will only end up at the node if it started directly on the eigenvector. Which is hard to do. Hence unstable.

For complex roots, eigenvalues are obtained as complex conjugate pairs (solving quadratics).

$\lambda_1, \lambda_2 \in \mathbb{C}$, **both have negative real parts.** Here, we have a stable spiral: the complex parts create an osscilatory term, and the negative eigenvalues mean that the eigenvectors point towards the origin.

$\lambda_1, \lambda_2 \in \mathbb{C}$, **both have positive real parts.** We have an unstable spiral.

$\lambda_1, \lambda_2 \in \mathbb{C}$, **both have zero real part** We now have concentric circles/ellipses, none of which decay to the centre.

This analysis has to be done carefully, as if the motion is too far away from a fixed point, then the linear stability analysis loses its validity.
We can do the same for higher dimensional systems, being careful that complex eigenvalues always occur in pairs (e.g. if there are 3 eigenvalues, then if one is found to be complex, there must be another). Visualising the classification becomes harder in more dimensions though.

3 Limit Cycles

'What does the system settle down to?'
A limit cycle is a single, unique isolated trajectory in phase space, and doesnt depend on the energy of the system.
That a limit cycle is isolated implies:

- If a limit cycle is stable, trajectories near to the cycle will fall towards it.
- If a limit cycle is unstable, trajectories near will move away.
- If a limit cycle is half-stable, trajectories inside the cycle will move towards, whereas those outside are repelled. Another case is where the opposite is true.

Limit cycles are hard to find analytically, unlike fixed points; but we can construct artificial examples:

Example If we have the system:

$$\frac{dr}{dt} = r(1 - r^2) \qquad \frac{d\theta}{dt} = 1$$

We can immediately solve for $\theta(t) = t + \theta(0)$. So, we see that the angle θ increases with time. Looking at the r equation, we see that $r = 0, 1$ are fixed points.
So, we have that θ increases with time; and that a fixed point is $r = 1$. So, a limit cycle is the unit circle.
Now, we must ask if the limit cycle is stable. To do this, we look at the sign of $f'(r^*)$, where $f(r) \equiv r(1 - r^2)$: standard linear stability analysis.
We find that $f'(0) = 1 > 0$, hence the origin is an unstable fixed point.
We find that $f'(1) = -2 < 0$, hence the unit circle is a stable limit cycle.

3.0.4 Poincare Oscillator

This is the system:

$$\dot{x} = -b(\sqrt{x^2+y^2}-a)-\omega y \tag{3.1}$$
$$\dot{y} = -b(\sqrt{x^2+y^2}-a)+\omega x \tag{3.2}$$

Where b, ω, a are all constants. Now, we introduce $\theta(t), r(t)$ via:

$$x(t) = r(t)\cos\theta(t) \tag{3.3}$$
$$y(t) = r(t)\sin\theta(t) \tag{3.4}$$

Now, we know that $r^2 = x^2 + y^2$. Hence, differentiating this with respect to time yeilds:

$$\begin{aligned} 2r\dot{r} &= 2x\dot{x} + 2y\dot{y} \\ \Rightarrow r\dot{r} &= -bx(\sqrt{x^2+y^2}-a) - \omega yx - by(\sqrt{x^2+y^2}-a) + \omega xy \\ &= -b(r-a)(x+y) \\ &= -b(r-a)(r\cos\theta + r\sin\theta) \\ &= -br(\cos\theta+\sin\theta)(r-a) \end{aligned}$$

Hence, we have an equation for $\dot{r}$:

$$\dot{r} = -b(\cos\theta+\sin\theta)(r-a) \tag{3.5}$$

Notice that immediately we can see that $r = a$ is some limit cycle.
Now, from (3.3), we can write down $\dot{x}$:

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ &= -b(\cos^2\theta + \sin\theta\cos\theta)(r-a) - r\dot{\theta}\sin\theta \end{aligned}$$

But, from (3.1), we have that

$$\dot{x} = -b(r-a) - \omega r\sin\theta$$

So:

$$\begin{aligned} -b(r-a) - \omega r\sin\theta &= -b(\cos^2\theta + \sin\theta\cos\theta)(r-a) - r\dot{\theta}\sin\theta \\ \Rightarrow b(r-a) + \omega r\sin\theta &= b(1-\sin^2\theta+\sin\theta\cos\theta)(r-a) + r\dot{\theta}\sin\theta \\ &= b(r-a) + b(r-a)(1-\sin^2\theta+\sin\theta\cos\theta)(r-a) + r\dot{\theta}\sin\theta \\ \Rightarrow 0 &= -b(-\sin^2\theta+\sin\theta\cos\theta)(r-a) - r\dot{\theta}\sin\theta + \omega r\sin\theta \end{aligned}$$

So, if $\sin\theta \neq 0$, we rearange to:

$$r\dot{\theta} = r\omega - b(\cos\theta - \sin\theta)(r-a) \tag{3.6}$$

Clearly, $r = a$ is a limit cycle, as $\dot{r} = 0$. Hence, we have that $\dot{\theta} = \omega \Rightarrow \theta(t) = \omega(t) + \theta_0$. Therefore, limit cycle is given by:

$$x(t) = a\cos(\omega t + \theta_0) \tag{3.7}$$
$$y(t) = a\sin(\omega t + \theta_0) \tag{3.8}$$

3.0.5 Van der Pol Oscillator

Here, the equation is:

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{3.9}$$

Where $\alpha > 0$. Notice that the damping term is non-linear.
We can scale items thus:

$$\tau \equiv \omega_0 t \tag{3.10}$$

$$\mu \equiv \frac{\alpha}{\omega_0} \tag{3.11}$$

Hence, we have that:

$$\frac{d^2x}{dt^2} = \omega^2 \frac{d^2x}{d\tau^2} \qquad \frac{dx}{dt} = \omega_0 \frac{dx}{d\tau}$$

Therefore, our equation becomes:

$$\frac{d^2x}{d\tau^2} - \mu(1 - x^2)\frac{dx}{d\tau} + x = 0$$

Now, if we turn this single second order equation into two single order equations, we have:

$$\frac{dx}{d\tau} = y \tag{3.12}$$

$$\frac{dy}{d\tau} = \mu(1 - x^2)y - x \tag{3.13}$$

Now, there is one fixed point: at the origin; and if linear stability analysis is done, it is found to be unstable. If $0 < \mu \leq 2$, we have an unstable spiral towards the centre, and for $\mu > 2$, an unstable node. Solutions can be plotted numerically to invesigate the behaviour and dependance on μ.

3.1 Ruling out Limit Cycles

Limit cycles are also referred to as closed orbits. We outline 2 methods of ruling out closed orbits:

3.1.1 Gradient Systems

Suppose we have the system:

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

Then the system is a gradient system, if there exists some $V(x, y)$, such that:

$$\begin{aligned} \frac{dx}{dt} = f &= -\frac{\partial V}{\partial x} \\ \frac{dy}{dt} = g &= -\frac{\partial V}{\partial y} \end{aligned}$$

To test if this exists, we see if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ holds. If it does, then a $V(x, y)$ exists.

Now, we claim that if a system has a potential V, then no closed orbit exists for the system.

Proof Suppose there exists a closed orbit in a system with a potential $V(x, y)$.
Then, after one circuit of the closed orbit, then change in V, ΔV, is zero.
Now, we have that:

$$\Delta V = \int_0^T \frac{dV}{dt}\, dt$$

Where T is the period of the orbit. We can write:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

As $V(x(t), y(t))$. Thus:

$$\frac{dV}{dt} = -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2$$

By the initial definition of the differentials of V. Hence, we see that $\Delta V \leq 0$:

$$\begin{aligned} \Delta V &= -\int_0^T \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt \\ &\leq 0 \end{aligned}$$

With $\Delta V = 0$ only for $\dot{x} = \dot{y} = 0$ i.e. at the fixed points.

Therefore, we have shown that if a V exists, it only has $\Delta V < 0$ for positions other than fixed points, and $\Delta V = 0$ only occurs at fixed points.
Therefore, no limit cycle exist for gradient systems.

This says nothing about systems which are not gradient systems.

Example Let $f = \dot{x} = \sin y$ and $g = \dot{y} = x \cos y$. We can easily show that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, therefore the system is a gradient system, therefore no limit cycles exist. (We can easily show that $V = -x \sin x$)

3.1.2 Liapunov Functions

A Liapunov function is some $\phi(x, y)$, for a system having a single fixed point at $x = x^*, y = y^*$, where ϕ has the following properties:

- $\phi(x^*, y^*) = 0$ and $\phi(x, y) > 0 \;\forall\; x, y \neq x^*, y^*$.
- $\frac{d\phi}{dt} < 0 \;\forall\; x, y \neq x^*, y^*$.

So, we have that $\phi(x, y)$ is some sort of bowl, where we may start anywhere up the sides, and as time progresses, we will always fall to the point $\phi(x^*, y^*) = 0$, regardless of initial conditions.
This is actually a statement of global stability:

$$\begin{aligned} \lim_{t\to\infty} x(t) &= x^* \\ \lim_{t\to\infty} y(t) &= y^* \end{aligned}$$

We can also prove that there are no limit cycles for a system for which a Liapunov function exists. To do this, we note that $\Delta\phi \leq 0$, when we have equality for $\frac{d\phi}{dt} = 0$ at the fixed point. In the definition of a Liapunov function we stated that $\frac{d\phi}{dt} < 0$.

$$\Delta\phi = \int_0^T \frac{d\phi}{dt} dt \leq 0$$

Now, the complicated thing is, that we must guess the form of $\phi(x, y)$, which can be hard!

Example Let us consider the system

$$\dot{x} = -x + 2y \qquad \dot{y} = -x - y - y^3$$

Let us guess that $\phi(x, y) = x^2 + y^2$. We find the fixed points of the system $\dot{x} = \dot{y} = 0$:

$$x = 2y \quad \Rightarrow \quad 3y = -y^3$$

So, we have either $y^* = 0$ or $(y^*)^2 = -3$. The last case is impossible, as we must have real fixed points.
Therefore, we have that the single fixed point is $(0, 0)$. We now must check our guess for ϕ as being the Liapunov function, by considering its properties.
We see that $\phi(x^*, y^*) = 0$, and for all $x, y \neq x^*, y^*$ we have that $\phi(x, y) > 0$.
The next condition is a little more complicated to show:

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} \\ &= 2x(-2 + 2y) + 2y(-x - y - y^3) \\ &= -2x^2 - 2y^2 + 2xy - 2y^4 \end{aligned}$$

It is not immediately obvious that this is less than zero. To show this:

$$\begin{aligned} (x - y)^2 &\geq 0 \\ \Rightarrow x^2 + y^2 - 2xy &\geq 0 \\ \Rightarrow x^2 + y^2 &\geq 2xy \end{aligned}$$

Hence, we can therefore state:

$$\begin{aligned} \frac{d\phi}{dt} &\leq -2x^2 - 2y^2 + x^2 + y^2 - 2y^4 \\ &\leq 0 \end{aligned}$$

And is only equal to zero for $x = y = 0$, the fixed point.
Therefore, we have shown that $\frac{d\phi}{dt} < 0$.
Therefore, $\phi(x, y) = x^2 + y^2$ is a Liapunov function for the system.
Therefore, due to the existence of the function, no limit cycle exists.

3.2 The Poincare-Bendixson Theorem

We can prove that a limit cycle exists, without having to find it.

Suppose R is some region in the 2D phase plane which contains no fixed points, and suppose that there is some trajectory C that is confined in R. Then, the PB theorem says that either:

- C is a closed orbit, or:
- C spirals towards a closed orbit as $t \to \infty$.

In either case, R contains a closed orbit.
Initially, there will probably be a fixed point within R, but we can just cut the fixed point out.
Another result of this theorem is that there cannot be any chaos in a 2D system.
This theorem only holds in 2D.
We do not prove the theorem.

The standard method for applying the theorem is to construct some trapping region R. We set things up, so that slope fields point inwards everywhere on the boundary of R.

Example Consider the system:

$$\begin{aligned} \dot{r} &= r(1-r^2) + \mu r \cos\theta \\ \dot{\theta} &= 1 \end{aligned}$$

Where $0 < \mu < 1$. So, does a limit cycle exist?
If $\mu = 0$, then $r = 1$ is a limit cycle; but this is found trivially, so we continue, with this artificial example to show how to use the PB theorem; let us now find if there is the possibility of a limit cycle for other values of μ.
Now, we see that there is only one fixed point at the origin $(0, 0)$, so we cut out the origin from our region. Hence, R contains no fixed points.
We construct our region out of two concentric circles, of radius r_{min}, r_{max}, centred on the origin. To get the slope fields to point into the region on the boundary, we impose that $\dot{r} > 0$ on $r = r_{min}$, and $\dot{r} < 0$ on $r = r_{max}$.
Considering r_{min}. We ask that:

$$(1-r^2) + \mu\cos\theta > 0 \quad \forall\,\theta$$

Or:

$$r^2 < 1 + \mu\cos\theta$$

But, $\cos\theta \geq -1$. So, if we put $r^2 < 1-\mu$, then $r^2 < 1 + \cos\theta$ for all θ. Hence:

$$r < (1-\mu)^{\frac{1}{2}}$$

And, if we take $r_{min} = 0.99(1-\mu)^{\frac{1}{2}}$, then we have satisfied our condition for $\dot{r} > 0$ on $r = r_{min}$, for all θ.
Simiarly, considering r_{max}, we ask that:

$$\begin{aligned} (1-r^2) + \mu\cos\theta < 0 \quad \forall\,\theta \\ \Rightarrow r^2 > 1 + \mu\cos\theta \end{aligned}$$

Buta again, we know that $\cos\theta \leq 1$. Therefore:

$$r > (1+\mu)^{\frac{1}{2}}$$

And, if we take $r_{max} = 1.001(1+\mu)^{\frac{1}{2}}$, then we have satisfied our condition for $\dot{r} < 0$ on $r = r_{max}$, for all θ.
Hence, we have constructed a trapping region, where in some $r_{min} < r < r_{max}$, we have shown that there is a limit cycle.

Notice, the PB theorem tells us that the dynamical possibilities in the 2D plane are limited to: escaping to infinity, go to a fixed point, go to a limit cycle.
The theorem has no analogue in higher dimensions.

However, in phase planes with dimension > 2, we have that trajectories may be attracted to a complex geometrical object: a strange attractor, which leads to chaotic behaviour.

4 Chaotic Dynamics

4.1 The Lorenz Equations

In 1963, Lorenz studied a very simple model of the atmosphere.
He modeled the ground being temperature T_1, and atmosphere being at T_2, where $T_1 > T_2$. Both surfaces are flat.
If ΔT is small, then you get conduction; but if ΔT is large, then convection occurs, with convection cells.
The fluid between the two model-plates can be described by 5 partial differential equations for density $\rho(\mathbf{x}, t)$, temperature $T(\mathbf{x}, t)$ and vecocity $\mathbf{v}(\mathbf{x}, t)$. To simplify the equations, Lorenz applied a Fourier expansion:

$$\begin{aligned} T(\mathbf{x}, t) &= \sum_{n,m} a_{mn}(t) \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi z}{\ell} \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{m\pi x}{L} \\ \rho(\mathbf{x}, t) &= \sum_{n,m} b_{mn}(t) \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi z}{\ell} \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{m\pi x}{L} \end{aligned}$$

And similar. To simplify, he threw away all modes, but the 2 lowest temperature and velocity modes. And he got out 3 ordinary differential equations for 3 of the time dependant amplitudes $a_{mn}(t)$ and $b_{mn}(t)$. If we now call the amplitudes $x(t), y(t), z(t)$; but not to be confused with a coordinate system! The equations are:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

Where σ, r, b are positive constants, and $r \propto T_1 - T_2 = \Delta T$. This system of equations are known as the Lorenz equations.
To start to analyse the system, we begin to look for fixed points. So:

$$\begin{aligned} x^* &= y^* b \\ x^*(r - 1 - z^*) &= 0 \\ (x^*)^2 &= bz^* \end{aligned}$$

So, we have 2 possibilities for fixed points:

- The origin $x^* = y^* = z^* = 0 \Rightarrow (0, 0, 0)$;
- $z^* = r - 1 \Rightarrow (x^*)^2 = (y^*)^2 = b(r - 1)$. We need $r \geq 1$ for real fixed points.

Thus, we have fixed points at $(0, 0, 0)$ and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$ for all $r > 1$.
For $r < 1$, we can setup a Liapunov functions to show that the origin is globally stable.
Take:

$$\phi(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$$

Under the Liapunov conditions that $\phi > 0$, apart from the origin where $\phi = 0$; and that $\frac{d\phi}{dt} < 0$.

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \\
&= \frac{2}{\sigma}x\sigma(y-x) + 2y(rx - y - xz) + 2z(xy - bz) \\
\Rightarrow \frac{1}{2}\frac{d\phi}{dt} &= xy - x^2 - y^2 \\
&= -\left(x - \frac{r+1}{2}y\right)^2 + \frac{(r+1)^2}{4}y^2 - y^2 - bz^2 \\
&= -\left(x - \frac{r+1}{2}y\right)^2 - \left(1 - \frac{(r+1)^2}{4}\right)y^2 - bz^2
\end{aligned}$$

Hence, we see that if

$$\frac{(r+1)^2}{4} < 1$$

then $\frac{d\phi}{dt} < 0$. That is:

$$\begin{aligned}
-1 < &\ \tfrac{r+1}{2} < 1 \\
\Rightarrow -2 < &\ r+1 < 2 \\
-3 < &\ r < 1
\end{aligned}$$

But, was also have that $r > 0$. Hence:

$$\frac{d\phi}{dt} < 0 \quad \forall \quad x, y, z \neq 0 \quad r < 1$$

Therefore, $\phi(x, y, z)$ is a Liapunov function for $r < 1$. That is, the origin is globally stable, and there are no limit cycles for $r < 1$.
Also, from the actual physics of the problem, $r < 1$ is conduction and $r > 1$ convection.
Therefore, $(0, 0, 0)$ represents conduction, and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ convection. The two convection solutions are denoted c^+ and c^-.
We do the proofs of the stability analysis in examples sheet 3, but:

- The fixed points c^+ and c^- are stable for $r > 1$, if $\sigma < b + 1$;
- If $\sigma > b + 1$, they are stable on the range

  $$1 < r < \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

  Where we define the above upper limit r_1;
- There are no stable limit cycles for $r < r_1$, or for values of r somewhat greater than r_1;
- All trajectories eventually enter and remain in a trapping region, so they are not repelled out to infinity.

The Lorenz system is then plotted as a projection onto the x, z-plane. Upon looking at the phase plane, we see a butterfly-wing patters, where the trajectories orbit in a spiral pattern indefinitely. In the actual 3D phase space, the attractor appears to be a set of thin surfaces (the 'wings'). It is actually an infinitely complex system of surfaces, with infinite surface area, and zero volume. It is these properties which lead to the conclusion that the attractor has dimension between 2 and 3. Numerical analysis finds the dimension to be 2.08. Thus, it is a strange attractor with fractal structure.

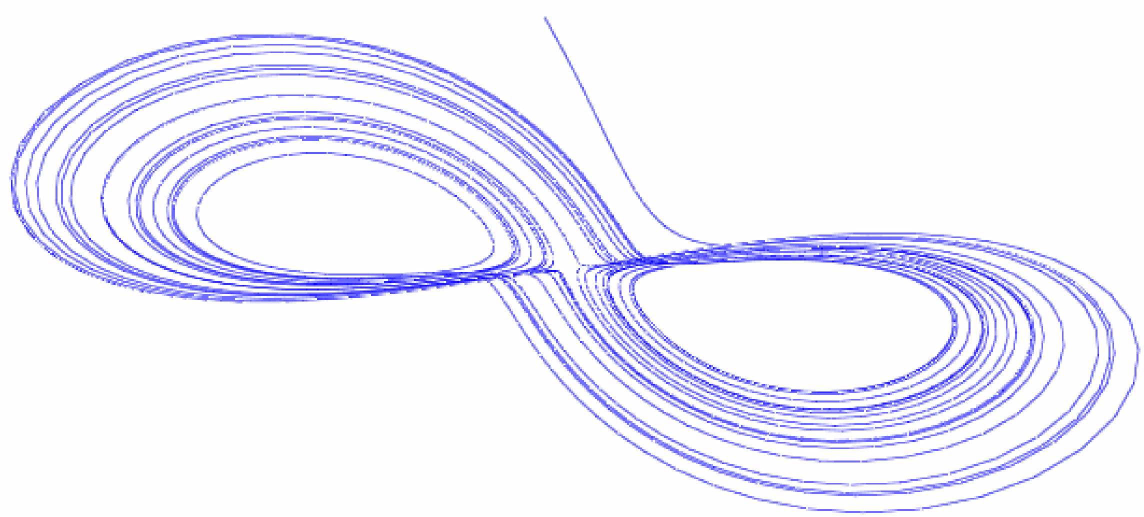


Figure 2: A view of the Lorenzian strange attractor. If the whole thing is viewed in its 3D phase space, it is seen that no two trajectories ever touch or cross; yet, there are infinitely many trajectories in a finite volume: a structure with fractional dimension.

4.2 Exponential Divergence of Nearby Trajectories

If we have 2 trajectories, which start very close to each other on the attractor, then they will rapidly diverge, each having very different futures. This obviously does not happen in non-chaotic systems.

Suppose $\mathbf{r}(t)$ is a point in phase space on the attractor at time t. And suppose $\mathbf{r} + \boldsymbol{\delta}(t)$ is a nearby point.
Suppose that the distance between the two points is $|\boldsymbol{\delta}(t)|^2$. Suppose that initially, the two points are a distance $|\boldsymbol{\delta}(0)|^2 = 10^{-10}$ apart (say). We would like to know how $|\boldsymbol{\delta}(t)|^2$ grows in time, so that we can have an estimation of errors, or how well we can predict things.
Numerical experimentation is done, and a dependance below is found:

$$|\boldsymbol{\delta}(t)|^2 = |\boldsymbol{\delta}(0)|^2 e^{Lt}$$

Where L is known as the Liapunov Exponent.
For the Lorenz attractor, it is numerically found that $L \approx 0.9$. L is a measure of how sensitive the system is to its initial conditions. Thus, we see that neighboring trajectories separate exponentially

fast. Note that for $L \leq 0$, there will not be any chaotic behavior. We need to formalise this definition slightly:

- For an n-dimensional system, there are n different Liapunov exponents

$$\delta_k(t) = \delta_k(0)e^{L_k t}$$

  Thus, the L we had before was the largest one, which is always the most important. So, for a 3D phase space, we will get 3 Liapunov exponents. We can visualise what is going on by thinking of a block of initial conditions, or a cube in phase space, at $t = 0$. Then, as the system evolves, points that started close together diverge or converge, according to the Liapunov exponent in a particular direction. So, our initial cube may end up as a squashed, bent rectangle. One side may now be thinner, whereas another may be streched over the entire attractor.

- L depends slightly on which trajectory we choose, so we must average over may trajectories.

Now, suppose we make an initial error $|\boldsymbol{\delta}(0)|$ in a measurement, and suppose that $L > 0$. So how do the true value and error change over time? How long does it take before our error becomes unacceptably large?
So, the discrepancy between our prediction and the true state grows to

$$|\boldsymbol{\delta}(t)| = |\boldsymbol{\delta}(0)|e^{Lt}$$

After some time t. If we state that our prediction becomes intolerably innacurate when $|\boldsymbol{\delta}(t)| \geq a$. Hence:

$$a \approx |\boldsymbol{\delta}(0)|e^{Lt}$$
$$\Rightarrow t \approx \frac{1}{L} \ln \frac{a}{|\boldsymbol{\delta}(0)|}$$

Hence we have an expression for finding the time t after which our prediction becomes intolerable by a.

Example Suppose that $|\boldsymbol{\delta}(0)| = 10^{-7}$, and $a = 10^{-3}$, then we find

$$\begin{aligned} t_1 &= \frac{1}{L} \ln \frac{10^{-3}}{10^{-7}} \\ &= \frac{1}{L} \ln 10^4 \\ &= \frac{4}{L} \ln 10 \end{aligned}$$

Now, suppose that we make our initial measurement a load more accurate: $|\boldsymbol{\delta}(0)| = 10^{-13}$, then we find:

$$t_2 = \frac{10}{L} \ln 10$$

Thus, we have that

$$\frac{t_2}{t_1} = 2.5$$

Thus, by making our initial measurement a million times more accurate, we only get results which are about 2.5 times better!

4.2.1 The 'Definition' of Chaos

'Chaos is aperiodic long term behaviour in a deterministic system that exhibits sensitive dependance on initial conditions.'

Unpacking this statement:
If a system is periodic, then there is a limit cycle. Thus, 'aperiodic long term' implies trajectories which do not settle down to fixed points, limit cycles etc as time goes to infinity.
That the system is deterministic, means that there is some smooth forcing term, no random or noisy inputs or parameters.
That the system has a sensitive dependance on initial conditions implies that initially nearby trajectories separate exponentially fast, and has some Liapunov exponent.

5 Non-Linear Maps

5.1 Linear and Quadratic Maps

Dynamical systems in which time is discrete (e.g $x_{n+1} = \cos x_n$, where $n = 0, 1, 2, \ldots$) are known as difference equations/recursion relations/iterated maps. We shall call them just maps.
1D maps have the form:

$$x_{n+1} = f(x_n)$$

Where x_0 is a given, and $f(x)$ is some given function. Here, we shall always consider $x_n \in \mathbb{R}$.

The simplest linear map is of the form:

$$x_{n+1} = \alpha x_n + \beta$$

Quadratic maps are of the form

$$x_{n+1} = \alpha + \beta x_n + \gamma x_n^2$$

If this is rescaled somehow, we get to a map known as the *logistic map*:

$$x_{n+1} = \lambda x_n(1 - x_n)$$

We now start to analyse the logistic map:

5.2 Simple Analysis of the Logistic Map

A plot of $f(x) = \lambda x(1-x)$ yields a simple looking parabola, and we only consider the portion of the plot on which $0 \leq x \leq 1$. Simple analysis gives a turning point where $f' = 0$, thus $\lambda(1 - 2x) = 0$, thus turning (maximum) point at $x = \frac{1}{2}$.

Notice, as long as $0 \leq \lambda \leq 4$, then the function $\lambda x_n(1-x_n)$ lies between 0 and 1, if x_n lies between 0 and 1.

To find fixed points of the logistic map, we want to find the point at which we get out what we put in, i.e. $x_{n+1} = x_n$. If we denote the fixed points x^*, as usual, then:

$$x^* = \lambda x^*(1 - x^*)$$

Notice, generally, a fixed point for a map is where $f(x^*) = x^*$.
Hence, for the logistic map, we find fixed points at:

$$\begin{aligned} x^* &= 0 \\ x^* &= 1 - \frac{1}{\lambda} \quad \lambda \neq 0 \end{aligned}$$

We can graph this as being the intersection of the graphs $f(x) = x$ and $f(x) = \lambda(1 - x)$.

We continue linear stability analysis, by perturbing the sytem by a small (hence linear) amount:

$$x_n = x^* + \hat{x}_n$$

Where $\hat{x}_n$ is some small perturbation. Hence, $x_{n+1} = f(x_n)$ becomes, under a Taylor expansion:

$$\begin{aligned} x^* + \hat{x}_{n+1} &= f(x^* + \hat{x}_n) \\ &= f(x^*) + \hat{x}_n f'(x^*) + \frac{1}{2}\hat{x}_n^2 f''(x^*) + \dots \end{aligned}$$

But, we have that $x^* = f(x^*)$; we also ignore quadratic perturbation terms; hence this reduces to:

$$\hat{x}_{n+1} = f'(x^*)\hat{x}_n$$

Or, defining $A \equiv f'(x^*)$:

$$\hat{x}_{n+1} = A\hat{x}_n$$

Hence, we see that we must have $|A| < 1$ for stability, as we want $\hat{x}_{n+1}$ to be smaller in magnitude than $\hat{x}_n$. Hence, we have the condition for stability:

$$\left|\frac{df}{dx}\right|_{x=x^*} < 1$$

Then x^* is some stable fixed point. The converse is also true:
If the following is true:

$$\left|\frac{df}{dx}\right|_{x=x^*} > 1$$

Then x^* is some unstable fixed point.

Thus, for the logistic map, we have that $f(x) = \lambda x(x - 1)$, hence:

$$\left|\frac{df}{dx}\right|_{x=x^*} = \lambda(1 - 2x^*)$$

At the fixed point $x^* = 0$, we thus have $\frac{df}{dx} = \lambda$, and hence $x^* = 0$ is a stable fixed point for $\lambda < 1$, and unstable on $\lambda > 1$.
At the fixed point $x^* = 1 - \frac{1}{\lambda}$, we have $\frac{df}{dx} = 2 - \lambda$. Now, this fixed point already has the condition that $\lambda > 1$. We can immediately see that for stability, $|2 - \lambda| < 1$. Hence, this fixed point is stable on $1 < \lambda < 3$, and unstable for $\lambda > 3$.

5.2.1 The 2-Cycle of the Logistic Map

So then, what happens if $\lambda > 3$, when both fixed points are unstable?
Numerical experiments have shown that the system settles down, and oscillates between two values p, q; whose value only depends on λ. That is:

$$\begin{aligned} p &= f(q) \\ q &= f(p) \\ \Rightarrow p &= f(q) = f(f(p)) \\ q &= f(p) = f(f(q) \end{aligned}$$

Thus, p and q are fixed points of $f(f(x))$. Thus, for the logistic map:

$$\begin{aligned} f(f(x)) &= \lambda f(x)(1-f(x)) \\ &= \lambda^2 x(1-x)(1-\lambda x(1-x)) \\ &= x \end{aligned}$$

Hence, we need to solve:

$$\begin{aligned} \lambda^2 x(1-x)(1-\lambda x(1-x)) = x \\ \Rightarrow x\{-\lambda^3 x^3 + 2\lambda^3 x^2 - \lambda^2(1+\lambda)x + (\lambda^2 - 1)\} = 0 \end{aligned}$$

This is a quintic equation, so has 4 roots. We already know that 2 of them are the previously found fixed points: $x^* = 0$ and $x^* = 1 - \frac{1}{\lambda}$. Thus, the equation has a factor of $x(\lambda x - x + 1)$. Hence, we can rewrite the quintic in terms of this factor:

$$x(\lambda x - x + 1)(-\lambda^2 x^2 + \gamma x - (\lambda + 1)) = 0$$

You can easily show, via linear independence, that $\gamma = \lambda(\lambda + 1)$. Hence, p, q can be found from the roots of the equation:

$$-\lambda^2 x^2 + \lambda(\lambda + 1)x - (\lambda + 1) = 0$$

Resulting in:

$$p, q = \frac{\lambda + 1}{2\lambda} \pm \frac{(\lambda + 1)^{1/2}(\lambda - 3)^{1/2}}{2\lambda}$$

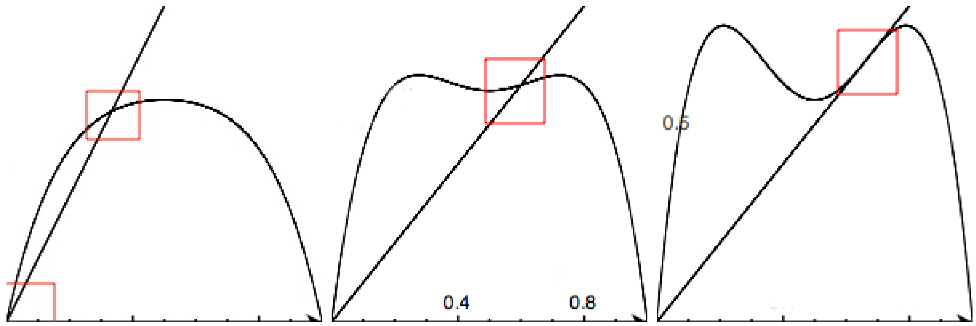
Only for $\lambda > 3$, as p, q becomes complex.

Now, we have a 2-cycle when $f(f(x)) = x$. We shall define:

$$g(x) \equiv f(f(x))$$

Hence, a 2-cycle when $g(x) = x$. We can graphically see what going on, by plotting $y = g(x)$ and $y = x$, and finding fixed points by their intersection.

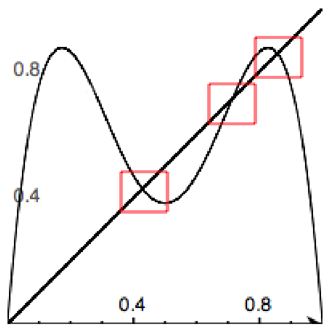
In the figure, you can see that for $\lambda = 3.5$ there are 3 fixed points. Infact, there are 3 fixed points for $\lambda = 3 + \varepsilon$, where ε is small. The fixed points are: the lower one is q, middle is $x^* = 1 - \frac{1}{\lambda}$ and top p.



(a) For $\lambda = 1.5$. Notice single well defined fixed point (other than at the origin)

(b) For $\lambda = 2.5$. Notice still single fixed point.

(c) For $\lambda = 3$. Notice that now the fixed point seems to be a tangent to $g(x)$.



(d) For $\lambda = 3.5$. Notice that now there are 3 fixed points.

Figure 3: Plots of various λ, showing how fixed points come about for the logistic map. Notice that there are 3 fixed points in (d). The red boxed just indicate fixed points.

Is the fixed point stable?
The stability test is:
Stable if:

$$\left|\frac{dg}{dx}\right|_{fp} < 1$$

Or unstable if:

$$\left|\frac{dg}{dx}\right|_{fp} > 1$$

Now, its not as hard as it sounds to calculate $\frac{dg}{dx}$. We can do this:

$$\begin{aligned}
\frac{d}{dx}g(x) &= \frac{d}{dx}f(f(x)) \\
&= f'(f(x))f'(x) \\
\Rightarrow \left.\frac{dg}{dx}\right|_p &= f'(f(p))f'(p) \\
&= f'(q)f'(p) \\
\Rightarrow \left.\frac{dg}{dx}\right|_q &= f'(f(q))f'(q) \\
&= f'(p)f'(q)
\end{aligned}$$

Hence, we see that the 2-cycle is stable if:

$$|f'(p)f'(q)| < 1$$

Now, $f'(x) = \lambda(1-2x)$, hence:

$$\begin{aligned}
f'(p) &= -1-(\lambda+1)^{1/2}(\lambda-3)^{1/2} \\
f'(q) &= -1+(\lambda+1)^{1/2}(\lambda-3)^{1/2} \\
\Rightarrow f'(p)f'(q) &= 4+2\lambda-\lambda^2
\end{aligned}$$

Hence, the 2-cycle is stable for:

$$-1 < 4+2\lambda-\lambda^2 < 1$$

Or, splitting this up, we must satisfy both:

$$\lambda^2-2\lambda-3>0 \qquad \lambda^2-2\lambda-5<0$$

The first can be shown to factorise to $(\lambda+1)(\lambda-3)>0$, hence providing us with the condition that $\lambda>3, \lambda<-1$.
The second gives us roots $\lambda = 1\pm\sqrt{6}$, hence we want $\lambda < 1+\sqrt{6} = 3.44$ for stability.
Therefore, the 2-cycle is stable on:

$$3 < \lambda < 1+\sqrt{6}$$

We can draw a bifurcation diagram, which has λ on the horizontal axis, and x on the vertical. It shows regions of stability. There will be one line for limit cycles, 2 for 2-cycles. We may propose that the 2-cycles at the end of their stability, give rise to 4-cycles.

5.3 Numerical Analysis of the Logistic Map

What happens just beyond $\lambda = 1+\sqrt{6}$? We find a 4-cycle, where the long term behaviour consists of jumping between 4 numbers, and are fixed points of $f(f(f(f(x))))$. This is a 16-th order polynomial, and is hence impossible to solve! Infact, this 4-cycle becomes unstable, and 'births' an 8-cycle, as λ is increased.
Infact, we find a succession of cycles of periods (n-cycle) $2, 4, 8, 16, 32, \ldots, 2^l, \ldots$

If we let Λ_l be the value of λ for which the 2^l-cycle is born.

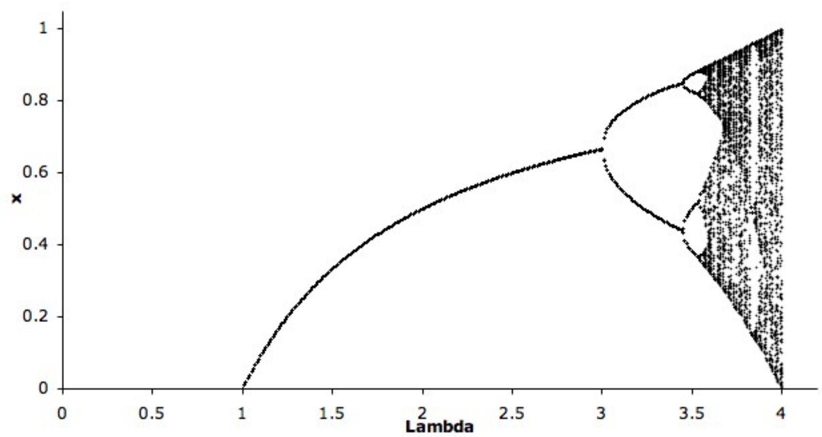


Figure 4: Bifurcation diagram for the Logistic map. The programme was run many times, to ensure that any transient behaviour had died away. Notice that a fixed point gives way to a 2-cycle, which births a 4-cycle, etc. These 2^n-cycles then eventually give way to chaos. Then, out of the chaos, starts a 3-cycles, which births a 6-cycle etc. Again, this ends in chaos. Eventually however, a 7-cycle, 14-cycle series is born. And so on, starting on the primes.

For example, we already have $\Lambda_1 = 3, \Lambda_2 = 1 + \sqrt{6} = 3.44$. And we can numerically compute others:

- $\Lambda_1 = 3$
- $\Lambda_2 = 3.44$
- $\Lambda_3 = 3.5440\dots$
- $\Lambda_4 = 3.5644\dots$
- $\Lambda_5 = 3.5687\dots$

etc.

In the 1970's Feigenbaum constructed the ratio:

$$\delta_n \equiv \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}}$$

And it was observed that the δ_n had values:

- $\delta_1 = 4.751$
- $\delta_2 = 4.657$
- $\delta_3 = 4.664$

- $\delta_4 = 4.666$

And appears to approach a constant $\equiv \delta$ as n becomes large:

$$\delta = \lim_{n\to\infty} \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}} = 4.6692016$$

It has also been shown that more-or-less any system has the same value of the Feigenbaum number δ! It is pretty much the analogy of pi to circles.

If you plot, on a straight line, λ, and mark off where the various Λ_n are, you can see how we can attempt to calculate Λ_∞, via:

$$\begin{aligned}
\Lambda_\infty - \Lambda_n &= (\Lambda_{n+1} - \Lambda_n) + (\Lambda_{n+2} - \Lambda_{n+1}) + \dots \\
&\approx (\Lambda_{n+1} - \Lambda_n) + \frac{(\Lambda_{n+1} - \Lambda_n)}{\delta} + \frac{(\Lambda_{n+1} - \Lambda_n)}{\delta^2} + \dots \\
&= (\Lambda_{n+1} - \Lambda_n)\left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + \frac{1}{\delta^3} + \dots\right) \\
&= (\Lambda_{n+1} - \Lambda_n)\frac{1}{1 - \delta^{-1}}
\end{aligned}$$

We also have that

$$\begin{aligned}
\Lambda_{n+1} - \Lambda_n &= \delta^{-1}(\Lambda_n - \Lambda_{n-1}) \\
&= \delta^{-2}(\Lambda_{n-1} - \Lambda_{n-2}) \\
&= \dots
\end{aligned}$$

Hence, we see that $\Lambda_\infty - \Lambda_n = C\delta^{-n}$, where C is some constant. i.e:

$$\Lambda_n = \Lambda_\infty - C\delta^{-n}$$

So, if we take the logarithm of this, we get:

$$\ln(\Lambda_n - \Lambda_\infty) = -n \ln \delta - \ln C$$

And, we can find numerically, that:

$$\Lambda_\infty = 3.5699457$$

So, what happens for $\Lambda > \Lambda_\infty$? A mixture of chaos and order is found: chaos, but with 'windows' of periodic behavior. There is, for example, a large window starting at $\lambda = 1 + 2\sqrt{2} = 3.82$ which begins with a stable 3-cycle, which then period doubles to 6, 12, ...

We get similar behavior in other non-linear maps:

The Sine map is defined thus:

$$x_{n+1} = \lambda \sin(\pi x_n)$$

Where, on $0 \le \lambda \le 1$, we have a return map so that $0 \le x_n \le 1$. The plot of $f(x) = \lambda \sin(\pi x)$ against x is pretty much the same as for the logistic map.
We see that the maximum is quadratic; that is, expansion is in terms of x^2.
We find that the bifurcation diagrams for the logistic and sine maps are amazingly similar!

When we say that there are windows of chaotic and periodic behavior, how do we define that there is chaotic behavior?
Well, for chaos, we must satisfy our previous definitions that it is aperiodic (2^∞ period is aperiodic), deterministic (no noisy inputs) and sensitive to initial conditions (Liapunov exponent is > 0). Lets find the Liapunov exponent:

Suppose we start with two points initially close together x_0 and $x_0 + \delta_0$.
If we apply the map n times, we get x_n and $x_n + \delta_n$.

The notation we shall use for applying a function $f(x)$ n times is:

$$f^n(x) = f(f(f(\ldots(x)\ldots)$$

Thus, our previous statement of 'applying the map n times' is equivalent to writing $f^n(x_0 + \delta_0) = x_n + \delta_n$ And we use our previous definition of the Liapunov exponent:

$$|\delta_n| = |\delta_0|e^{Ln}$$

Note, we have $L > 0$ for chaos and $L < 0$ for a limit cycle. Hence, upon rearrangement, we have:

$$\begin{aligned} L &= \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right|_{x=x_0} \\ &= \frac{1}{n}\ln\left|\frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0}\right|_{x=x_0} \end{aligned}$$

Notice, under the limit of $\delta_0 \to 0$, this is just the definition of the differential at $f^n(x)$:

$$L = \frac{1}{n}\ln\left|\frac{df^n}{dx}\right|_{x=x_0}$$

Now, this differential is not as hard to compute as we may immediately think. Consider what we had previously:

$$\left.\frac{df^2}{dx}\right|_{x=x_0} = f'(x_1)f'(x_0)$$

We can see that:

$$\left.\frac{df^3}{dx}\right|_{x=x_0} = f'(x_2)f'(x_1)f'(x_0)$$

And that:

$$\left.\frac{df^n}{dx}\right|_{x=x_0} = f'(x_{n-1})f'(x_{n-2})\ldots f'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

Hence, we stick this into the Liapunov exponent expression:

$$\begin{aligned} L &= \frac{1}{n}\ln\left|\frac{df^n}{dx}\right|_{x=x_0} \\ &= \frac{1}{n}\ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)| \end{aligned}$$

Where we have that the x_i are the values jumped between in an n-cycle.

And for the logistic map, plotting L/λ, we can see where $L > 0$, so we can see where there are periods of chaos.

5.4 Higher Dimensional Maps

In 1D we had:

$$x_{n+1} = f(x_n)$$

In 2D we have:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned}$$

And in ND, we have:

$$\begin{aligned} (x_1)_{n+1} &= f_1((x_1)_n, (x_2)_n, \ldots, (x_N)_n) \\ (x_2)_{n+1} &= f_2((x_1)_n, (x_2)_n, \ldots, (x_N)_n) \\ &\vdots \\ (x_N)_{n+1} &= f_N((x_1)_n, (x_2)_n, \ldots, (x_N)_n) \end{aligned}$$

Notation gets a little tricky here, so we'll stick to 2D, but all arguments apply equally as well to ND.

We'll work with something called the Henon Map:

$$\begin{aligned} f(x, y) &= 1 - ax^2 + y \\ g(x, y) &= bx \qquad\qquad a, b \neq 0 \end{aligned}$$

We choose b so that:

$$0 < b < 1$$

Now, to find the fixed points of the system.

Fixed points are such that:

$$\begin{aligned} f(x^*, y^*) &= x^* \\ g(x^*, y^*) &= y* \end{aligned}$$

That is:

$$\begin{aligned} y^* &= bx^* \\ x^* &= 1 - a(x^*)^2 + y^* \\ \Rightarrow 0 &= a(x^*)^2 - bx^* + x^* - 1 \\ \Rightarrow 0 &= a(x^*)^2 + (1-b)x^* - 1 \\ \Rightarrow x^* &= \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a} \end{aligned}$$

We require that $(1-b)^2 + 4a \geq 0$ for real fixed points. Therefore, if we define $a_0 \equiv -\frac{1}{4}(1-b)^2$, we only get fixed points for:

$$a > a_0$$

A 2-cycle of the system is such that:

$$x_{n+2} = x_n \qquad y_{n+2} = y_n$$

So:

$$\begin{aligned} x_{n+2} &= 1 - a_{n+1}^2 + y_{n+1} \\ &= 1 - a(1 - ax_n^2 + y_n)^2 + bx_n \\ y_{n+2} &= bx_{n+1} \\ &= b(1 - ax_n^2 + y_n) \end{aligned}$$

Thus, to get a 2-cycle, we have it at the fixed points:

$$\begin{aligned} x^* &= 1 - a[1 - a(x^*)^2 + y^*]^2 + bx^* \\ y^* &= b[1 - a(x^*)^2 + y^*] \end{aligned}$$

Which, after a bit of algebra, and substituting in for y^*, results in:

$$a^3(x^*)^4 - 2a^2(x^*)^2 + (1-b)^3 x^* + (a - [1-b]^2) = 0$$

Now, the previously found fixed points must also satisfy this equation. The fixed point quadratic must therefore be a factor to this quintic. Hence:

$$[a(x^*)^2 + (1-b)x^* - 1][a^2(x^*)^2 + \gamma x^* - [a - (1-b)^2]] = 0$$

We can find the γ term by looking at all the $(x^*)^3$ terms, and putting their sum to zero (as no $(x^*)^3$ term in the original quintic). Thus:

$$a\gamma(x^*)^3 + a^2(1-b)(x^*)^3 = 0$$

Resulting in $\gamma = -a(1-b)$. Hence, the 2-cycles satisfy:

$$a^2(x^*)^2 - a(1-b)x^* - [a - (1-b)^2]$$

Which has solutions:

$$x^* = \frac{(1-b) \pm \sqrt{4a - 3(1-b)^2}}{2a}$$

Where the positive root corresponds to the point p, and negative q. Again, these points only exist if the argument of the square root is positive:

$$a > a_1 \qquad a_1 \equiv \frac{3}{4}(1-b)^2$$

Remember, we also only have fixed points for $a > a_0$.

Thus, we have computed the values between which x hops: p and q. There will be equivalents for y.

5.4.1 Stability of Fixed Points

Suppose we have the general 2D map:

$$\begin{aligned}
x_{n+1} &= f(x_n, y_n) \\
y_{n+1} &= g(x_n, y_n)
\end{aligned}$$

Let us perturb around the fixed point x^* by some small amount $\hat{x}_n$:

$$\begin{aligned}
x_n &= x^* + \hat{x}_n \qquad x_{n+1} = x^* + \hat{x}_{n+1} \\
y_n &= y^* + \hat{y}_n \qquad y_{n+1} = y^* + \hat{y}_{n+1}
\end{aligned}$$

Now, if we Taylor expand around each fixed point:

$$\begin{aligned}
x^* + \hat{x}_{n+1} &= f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{FP} \hat{x}_n + \left.\frac{\partial f}{\partial y}\right|_{FP} \hat{y}_n + \dots \\
y^* + \hat{y}_{n+1} &= g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{FP} \hat{x}_n + \left.\frac{\partial g}{\partial y}\right|_{FP} \hat{y}_n + \dots
\end{aligned}$$

Where we shall ignore any quadratic terms. Notice, $x^* = f(x^*, y^*)$ and $y^* = g(x^*, y^*)$ by definition, so that they cancel. This then leaves us to write this in matrix form:

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{FP} \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix}$$

That is, the Jacobian:

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix}$$

Where the elements have been evaluated at the fixed point.

Now, we can expand a matrix in terms of its eigenvectors:

$$\begin{aligned}
\begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix} &= \sum_k c_n^k \boldsymbol{e}^k \\
\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} &= \sum_k c_{n+1}^k \boldsymbol{e}^k
\end{aligned}$$

Where $\boldsymbol{e}^k$ is the eigenvector of the Jacobian corresponding to the eigenvalue μ^k. Where the sum over k is just for $k = 1, 2$. The reason we have notated things like this is to show the generality of the method. We obviously have the eigenvalue equation:

$$J\boldsymbol{e}^k = \mu^k \boldsymbol{e}^k$$

So, substituting this in:

$$\begin{aligned}
\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} &= J \sum_k c_n^k \boldsymbol{e}^k \\
&= \sum_k c_n^k J \boldsymbol{e}^k \\
&= \sum_k c_n^k \mu^k \boldsymbol{e}^k \\
\Rightarrow \sum_k c_{n+1}^k \boldsymbol{e}^k &= \sum_k c_n^k \mu^k \boldsymbol{e}^k
\end{aligned}$$

From which we can see:

$$c_{n+1}^k = c_n^k \mu^k \qquad \forall\, k$$

Now, notice that for stability, we want $|\mu^k| < 1$; so that we get 'sucked in' when we increase up the iterates. That is, we want all eigenvalues of the Jacobian to be less than 1. If eigenvalues are complex, then its obviously the magnitude of the number $\mu\mu^* < 1$.

Now, if you remember, for the case of differential systems (continuous time) we required that $\Re(\lambda^k) < 0$, whereas now we want $\mu^k < 1$. This is because of the nature of the way we formulated the eigenequations.
Now, we have that $c_n = \mu^n c_0$, where before we have $(e^\lambda)^t$. We actually required that $e^\lambda < 1$, which implied $\lambda < 0$. Thus no contradiction in definitions.

So, for the Henon Map:

$$f(x,y) = 1 - ax^2 + y \qquad g(x,y) = bx$$

Hence:

$$J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}_{FP} = \begin{pmatrix} -2ax^* & 1 \\ b & 0 \end{pmatrix}$$

Hence, we find its eigenvalues from:

$$\begin{aligned}
\begin{vmatrix} -2ax - \mu & 1 \\ b & -\mu \end{vmatrix} &= 0 \\
\Rightarrow (-2ax^* - \mu)(-\mu) - b &= 0 \\
\Rightarrow \mu^2 + 2ax^*\mu - b &= 0 \\
\Rightarrow \mu &= -ax^* \pm \sqrt{(ax^*)^2 + b}
\end{aligned}$$

Hence, for each fixed point, we will have 2 values of μ, the eigenvalues of the Jacobian: $\mu^\pm$.

So, we have two fixed points:

$$\begin{aligned}
x_+^* &= \frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2a} \\
x_-^* &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a}
\end{aligned}$$

So, lets find the stability of the x_+^* fixed point, at the μ^+ eigenvalue. We do this by pre-supposing that the fixed point is stable, and seeing if everything holds. So, we suppose that $\mu^+ < 1$:

$$\begin{aligned} -ax_+^* + \sqrt{(ax_+^*)^2 + b} &< 1 \quad (5.1) \\ \Rightarrow \sqrt{(ax_+^*)^2 + b} &< 1 + ax_+^* \quad (5.2) \end{aligned}$$

So, we need to check that $1 + ax_+^* > 0$, so that we can square the above expression to get rid of the square-root. We do this by noting that the above expression for x_+^* can be written:

$$\begin{aligned} 2ax_+^* &= b - 1 + \sqrt{(1-b)^2 + 4a} \\ \Rightarrow 2ax_+^* + 2 &= b + 1 + \sqrt{(1-b)^2 + 4a} \end{aligned}$$

Hence, as the above square-root is real, and $b > 0$, we have that $1 + ax_+^* > 0$. Hence, squaring (5.2), gives:

$$\begin{aligned} (ax_+^*)^2 + b &< 1 + 2ax_+^* + (ax_+^*)^2 \\ \Rightarrow 2ax_+^* + (1-b) &> 0 \\ \Rightarrow \sqrt{(1-b)^2 + 4a} &> 0 \end{aligned}$$

Where the last step has been done by inserting the expression for x_+^*. This square root is always greater than zero. Hence we have shown that the pre-supposition of $\mu^+ < 1$ is true.

We can similarly show that $\mu^- < 1$. We can also show that $\mu^+ > -1$ and $\mu^- > -1$ if $a < a_1$, where:

$$a_1 \equiv \frac{3}{4}(1-b)^2$$

And also that $\mu^- < -1$ if $a > a_1$. Infacts, a_1 has been previously derived as the value for which 2-cycle starts to exist.
Thus, we have shown when the fixed point ceases to be stable, and have shown that it is at the same point at which the 2-cycle is born.

5.4.2 Stability of 2-Cycles

We have that a 2-cycle is defined as:

$$\begin{aligned} x_{n+2} &= f(x_{n+1}, y_{n+1}) = f(f(x_n, y_n), g(x_n, y_n)) \\ y_{n+2} &= g(x_{n+1}, y_{n+1}) = g(f(x_n, y_n), g(x_n, y_n)) \end{aligned}$$

So we return back to the same point after 2 iterations.

We proceed through stability analysis as before, by perturbing slightly around a fixed point:

$$x_n = x^* + \hat{x}_n \qquad y_n = y^* + \hat{y}_n$$

And, same as before, we end up writing a Jacobian. If the positions of the 2-cycle are at $\boldsymbol{p}$ and $\boldsymbol{q}$, then:

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{\boldsymbol{r}=\boldsymbol{p}} \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix}$$

This is done once for each fixed point:

$$\begin{pmatrix} \hat{x}_{n+2} \\ \hat{y}_{n+2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{\boldsymbol{r}=\boldsymbol{q}} \begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix}$$

Hence, if we combine these, we get:

$$\begin{pmatrix} \hat{x}_{n+2} \\ \hat{y}_{n+2} \end{pmatrix} = J_{\boldsymbol{r}=\boldsymbol{p}} J_{\boldsymbol{r}=\boldsymbol{q}} \begin{pmatrix} \hat{x}_{n} \\ \hat{y}_{n} \end{pmatrix}$$

So, to investigate the stability of the 2-cycle, we need to study the eigenvalues of $J^{(2)} \equiv J_{\boldsymbol{r}=\boldsymbol{p}} J_{\boldsymbol{r}=\boldsymbol{q}}$. Notice, this is a generalisation of the previous $|f'(p)f'(q)| < 1$. Notice that this also easily generalises to N dimensions.

So, for the Henon map:

$$J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

Thus:

$$J^{(2)} = \begin{pmatrix} -2aq & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} -2ap & 1 \\ b & 0 \end{pmatrix}$$

Notice, in the example of the Henon map, the positions of the 2-cycle are only in terms of the x-coordinate, and not y. This procedure generalises out to encompass this case. Thus:

$$J^{(2)} = \begin{pmatrix} 4a^2pq + b & -2aq \\ -2abp & b \end{pmatrix}$$

The eigenvalues μ of which can be found from the characteristic equation:

$$\mu^2 + \mu(-4a^2pq - 2b) + b^2 = 0$$

And, putting in the previously found expressions for p, q gives:

$$\mu^2 - 2\mu(2a - 2(1-b)^2 - b) + b^2 = 0$$

Again, after huge amounts of algebra, we find that the 2-cycle is stable (i.e. $\mu < 1$) for $a < a_2$ where:

$$a_2 \equiv \frac{5b^2 - 6b + 5}{4}$$

And unstable on $a > a_2$.

So, in our analysis of the stability of fixed points and 2-cycles and further, we see that as a period of stability ends, another of double-period begins. Again, if we look at a Feigenbaum-type number: If we compute the interval in which a 4-cycle exists, and divide it by the interval within which a 8-cycle exists, we compute a δ. Generally:

$$\delta_n = \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}}$$

We find that we approach the standard $\delta = 4.6692016$, which corresponds to a 2^{∞}-period object at a_{∞}.

6 Fractals

6.1 How Long is the Coastline of Britain?

This is the standard way of introducing fractals.

Suppose we want to measure the length ℓ of the coast around britain, We can do this using a compass of certain length scale/resolution; we denote the length scale used η. If we use a large length-scale, say, 500km, we get a value of 2600km. However, if we use a more accurate, smaller length scale, η =100km, we get $\ell = 3800$km. If we carry on increasing the resolution, decreasing the length scale used to measure the coastline, we find that ℓ increases. If $\eta = 17$km, then $\ell = 8640$km. In principle, as $\eta \to 0$ we should get $\ell \to \infty$. So, the coastline length depends on the scale in which it is measured, as it has an irregular structure. Hence, we say it actually has no 'proper' length. This example dosent work on all length scales of course, but it does illustrate the ideas involved.

Let η be the length scale at which measurements are taken, and call the coastline length $\ell(\eta)$. If $\ell(\eta)$ is plotted against η on a log-log scale, we find a constant slope. That suggests a dependance thus:

$$\ell(\eta) = C\eta^\alpha$$

Where C, α are constants. For the british coastline problem, we find $\alpha = -0.36$. Another property of the coastline is that if a bit of it is drawn, it is impossible to determine on which length scale it is based upon.

Thus, we say that the coastline of britain is a *fractal*. We define the term fractal thus:

A fractal is an object with comple structure on all scales and which has some degree of self similarity.

An example of an artificially constructed fractal is known as the *von Koch curve*.

6.1.1 The von Koch Curve

Begin with a straight line segment, of length L. That is the first iteration, and is known as S_0. Now, take 4 lines of length $\frac{L}{3}$, and arrange them into an equilateral triangle as in the figure. This is now S_1, where the total length of the shape is $\frac{4}{3}L$. Then do the process again, where each segment has length $\frac{1}{9}L$; The total length of the shape is now thus $\frac{16}{9}L = \frac{4^2}{3^2}L$, and is called S_2. This process is now repeated, essentially indefinitely. The von Koch curve itself is that for S_∞.

So, the length of S_n is given by:

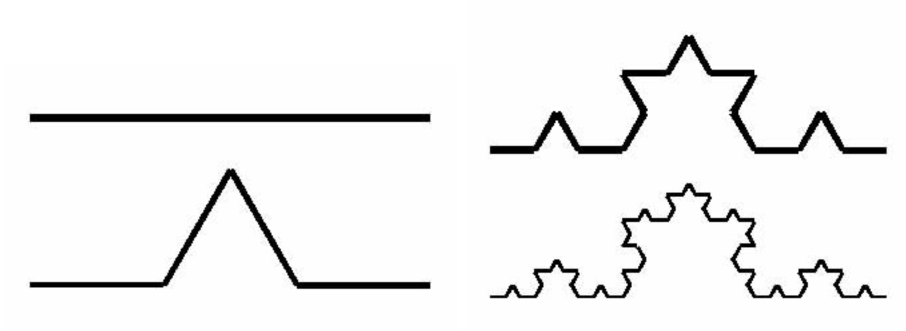
$$L_n = \frac{4^n}{3^n}\ell$$

Hence, the ratio of length S_n to S_0 is thus:

$$\left(\frac{4}{3}\right)^n$$

Hence, the length of the von Koch curve S_∞ is:

$$\lim_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty$$



(a) Showing S_0 and S_1 of the von Koch curve.

(b) Showing S_2 and S_3.

Figure 5: Iterates on the way to becoming a von Koch curve.

We hence have a shape which is infinitely longer than a line, but its not quite an area. A fractal. Notice, for the von Koch curve, if we zoom in on any portion of it, it looks exactly the same as the un-zoomed version.

6.2 Similarity Dimension

Suppose we have a square S_0. We then cut it into quarters, so that there are 4 mini-squares: 4 copies of the original, each of which has a length of half the original. This is S_1. Suppose we do it again; we get 16 mini-squares, each reduced in size by a factor of 2: S_2.
Each time, we get m copies of the original, reduced by a factor of r in length. So here, $m = 4$ and $r = 2$. If we do this, but starting by thirding everything, we have $r = 3, m = 9$.
In general, $m = r^2$. We see that 2 is the dimension of space. Thus, for a cube, $m = r^3$. For a hypercube in d-D, we have $m = r^d$. Hence:

$$\begin{aligned} m &= r^d \\ \Rightarrow \ln m &= d \ln r \\ \Rightarrow d &= \frac{\ln m}{\ln r} \end{aligned}$$

The quantity d is defined as the *similarity dimension.*

For the von Koch curve, we had 4 copies, with factor of 3 reduction in length. Hence, $r = 3, m = 4$. Thus, for the von Koch curve, the similarity dimension is:

$$d = \frac{\ln 4}{\ln 3} = 1.261$$

Which is obviously a non-integer dimension.

So, we now ask the question: How is the similarity dimension d related to α, the exponent appearing in $L(\eta)$?

We initialy state that:

$$\eta = \frac{\ell}{r^n} \quad \Rightarrow \quad r^{-n} = \frac{\eta}{\ell}$$

Where η is the scale relevant to the n^{th} segment. Thus, taking logs:

$$\begin{aligned} \ln r^{-n} &= \ln \frac{\eta}{\ell} \\ \Rightarrow n &= -\frac{\ln\left(\frac{\eta}{\ell}\right)}{\ln r} \end{aligned}$$

Now, in the von Koch problem, we had that the length L_n of the n^{th} iterate is related to the original length ℓ via:

$$\begin{aligned} L_n &= \left(\frac{m}{r}\right)^n \ell \\ \Rightarrow \ln\left(\frac{L_n}{\ell}\right) &= n \ln\left(\frac{m}{r}\right) \\ &= -\frac{\ln\left(\frac{\eta}{\ell}\right)}{\ln r} \ln\left(\frac{m}{r}\right) \\ &= -\ln\left(\frac{\eta}{\ell}\right) \frac{\ln m - \ln r}{\ln r} \\ &= -\ln\left(\frac{\eta}{\ell}\right)\left(\frac{\ln m}{\ln r} - 1\right) \\ &= -\ln\left(\frac{\eta}{\ell}\right)[d-1] \\ \Rightarrow \frac{L_n}{\ell} &= \left(\frac{\eta}{\ell}\right)^{1-d} \end{aligned}$$

Or, rewriting as:

$$L(\eta) = C\eta^{\alpha} \quad C \equiv \ell^d \quad \alpha \equiv 1-d$$

So, the equivalent exponent for the von Koch curve is $\alpha = -0.261$. The dimension of the british coastline is thus $d = 1 - \alpha = 1.36$.

Other self-similar fractals:

6.2.1 The Cantor Set

Start with $S_0 = [0, 1]$ and remove its middle third, the interval $(\frac{1}{3}, \frac{2}{3})$, where square brackets include the extremes, curved do not. This produces S_1.

This process is the repeated on each line that is left, untill S_∞, the Cantor Set.

Thus, we produce 2 copies ($m = 2$), each being scaled by a factor of 3 ($r = 3$). Hence, we can calculate its fractal dimension:

$$d = \frac{\ln 2}{\ln 3} = 0.63$$

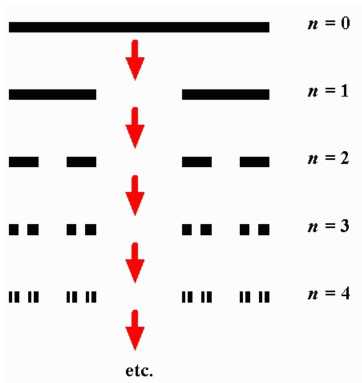
Hence, we see that its less than a line, more than a point.

6.2.2 The von Koch Snowflake

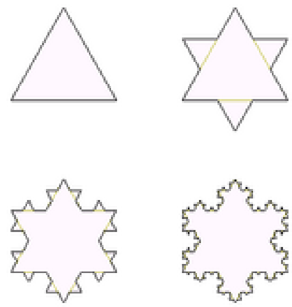
This is similar to the 'curve', but begins on an equilateral triangle

6.2.3 The Sierpinski Gasket

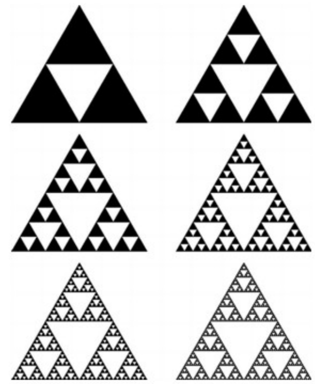
Here, we take out the middle section of a triangle, and keep repeating. Thus, we have $m = 3$ copies, and $r = 2$ scale factor, to give $d = 1.5849$ as the dimension of the Gasket. The Sierpinski carpet is constructed similarly, but with squares, and has $d = 2.09$.



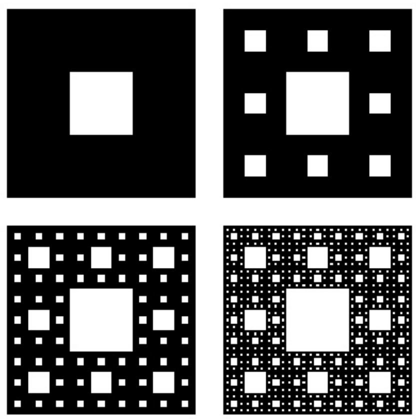
(a) The Cantor Set.



(b) The von Koch snowflake.



(c) The Sierpinski Gasket.



(d) The Sierpinski Carpet.

Figure 6: Showing how various fractals are constructed.

7 Further Aspects of Chaotic Dynamics

7.1 How do Volumes Evolve in Phase-Space?

Suppose we have the 3D system:

$$\begin{aligned} \dot{x} &= f_1(x,y,z) \\ \dot{y} &= f_2(x,y,z) \\ \dot{z} &= f_3(x,y,z) \end{aligned}$$

Then we can write it as:

$$\dot{\boldsymbol{r}} = \boldsymbol{f}(x,y,z)$$

Where:

$$\boldsymbol{r} \equiv x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \boldsymbol{f} \equiv f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$$

Now, suppose we locate a bit of phase space, with volume V, at time t, which has a particular set of initial conditions. This is a 'blob' of initial conditions. So, how does the volume of the blob change in time?

Suppose we let the volume $V(t)$ evolve for a time δt, to $V(t+\delta t)$.
We proceed by considering a small patch of area dA on the surface of volume $V(t)$. Let this patch of area move a distance δT (thickness) in time δt. Thus, δT will be the velocity the patch moved multiplied by the time taken δt. The normal component of the velocity of the patch is given by $\boldsymbol{v}\cdot\boldsymbol{n} = \dot{\boldsymbol{r}}\cdot\boldsymbol{n} = \boldsymbol{f}\cdot\boldsymbol{n}$. Hence:

$$\delta T = \boldsymbol{f}\cdot\boldsymbol{n}\delta t$$

Thus, the volume that the patch sweeps out in time δt is just $\delta T\, dA$:

$$\delta T\, dA = \boldsymbol{f}\cdot\boldsymbol{n}\delta t\, dA$$

So, the total volume of the blob after time δt is $V(t+\delta t)$, which is the same as the sum of the initial volume $V(t)$ and the volume swept out by all the little patches moving:

$$\begin{aligned} V(t+\delta t) &= V(t) + \int_S \boldsymbol{f}\cdot\boldsymbol{n}\delta t\, dA \\ \Rightarrow \frac{V(t+\delta t) - V(t)}{\delta t} &= \int_S \boldsymbol{f}\cdot\boldsymbol{n}\, dA \end{aligned}$$

Now, the LHS, under the limit of $\delta t \to 0$ is just the differential. Hence:

$$\frac{dV(t)}{dt} = \int_S \boldsymbol{f}\cdot\boldsymbol{n}\, dA$$

The divergence theorem states:

$$\int_S \boldsymbol{f}\cdot\boldsymbol{n}\, dA = \int_V \nabla\cdot\boldsymbol{f}\, dV$$

Hence, we see that the rate of change of a volume in phase space is given by the integral of the divergence over a volume:

$$\frac{dV}{dt} = \int_V \nabla\cdot\boldsymbol{f}\, dV \qquad (7.1)$$

This has thus far been for a 3D system. If the system is 2D, volumes become areas, and areas become lines. Thus:

$$\frac{dA}{dt} = \oint_C \boldsymbol{f} \cdot \boldsymbol{n} \, d\ell$$

For n-dimensions, the divergence is just $\sum_i \frac{\partial f_i}{\partial x_i}$. Hence, we have:

$$\frac{dV}{dt} = \int_V \sum_i \frac{\partial f_i}{\partial x_i} \, dV$$

Where a volume is defined in terms of the dimension of the system.

Example Consider the Lorenz equations:

$$\begin{aligned} \dot{x} &= f_1 = \sigma(y - x) \\ \dot{y} &= f_2 = rx - y - xz \\ \dot{z} &= f_3 = xy - bz \end{aligned}$$

Thus, we have that $\nabla \cdot \boldsymbol{f}$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(\sigma + b + 1)$$

Thus, the divergence is actually independent of xy, z. Hence:

$$\begin{aligned} \frac{dV}{dt} &= \int_V \nabla \cdot \boldsymbol{f} \, dV \\ &= -(\sigma + b + 1) \int_V dV \\ &= -(\sigma + b + 1) V(t) \\ \Rightarrow \frac{dV}{V(t)} &= -(\sigma + b + 1) dt \\ \Rightarrow V(t) &= V(0) e^{-(\sigma + b + 1)} \end{aligned}$$

Thus, we see that a volume will shrink to zero as time progresses; that is, the blob of initial conditions all settle down to some limiting set, be that a fixed point, limit cycle or strange attractor.

Example Consider a system with even dimension $n = 2m$. Suppose further that the system is conservative. Thus, we can use Hamiltonian mechanics, with the first m coordinates as being the generalised postions q_i and the last m as being generalised momenta p_i:

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \quad \alpha = 1, 2, \ldots, m$$

Hence, we have that:

$$\begin{aligned} f_\alpha &= \frac{\partial H}{\partial p_\alpha} \\ f_{\alpha+m} &= -\frac{\partial H}{\partial q_\alpha} \end{aligned}$$

Now, this is quite subtle: the total function we take the divergence of is actually:

$$f = \sum_{\alpha=1}^{\alpha=m} f_\alpha + \sum_{\alpha=m+1}^{\alpha=2m} f_{\alpha+m}$$

It is important to note that coordinates x_α for $\alpha = 1, \dots m$ are the generalised coordinates q_α, and are generalised momenta p_α for $\alpha = m+1, \dots 2m(=n)$.
So, when we take the divergence:

$$\begin{aligned} divf &= \sum_i \frac{\partial f_i}{\partial x_i} \\ &= \sum_{\alpha=1}^{m} \frac{\partial f_\alpha}{\partial x_\alpha} + \sum_{\alpha=m+1}^{2m} \frac{\partial f_{\alpha+m}}{\partial x_\alpha} \\ &= \sum_{\alpha=1}^{m} \frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} + \frac{\partial}{\partial p_\alpha}\left(-\frac{\partial H}{\partial q_\alpha}\right) \\ &= \sum_{\alpha=1}^{m} \frac{\partial^2 H}{\partial q_\alpha \partial p_\alpha} - \frac{\partial^2 H}{\partial p_\alpha \partial q_\alpha} \\ &= 0 \end{aligned}$$

Therefore, we have that volumes in phase space do not change for conservative systems. Thus, conservative systems do not have objects like fixed points, limit cycles, strange attractors.
This is known as Liouville's theorem.

7.2 Folding and Stretching

Properties of trajectories in phase space:

- Dont cross, because there is a unique solution to a particular set of initial conditions. This is true for all systems;
- Lie on attractors of zero volume as $t \to \infty$, for non-conservative systems;
- Separate exponentially fast from their neighbours (if chaotic, positive Liapunov exponent), at least initially. Yet, they remain confined to a bounded region of phase space; at least for those on strange attractors. Hence, at the edge of a bounded region, trajectories fold back on each other; this is where chaotic behaviour comes from.

Trajectories repeatedly fold and stretch. A volume of inital conditions changes its shape, with some surfaces being stretched, some compressed; depending on the Liapunov exponent in that particular direction.

Infact, if we look at some cross-sections of a partcular folding, we can see the Cantor set emerge!

7.3 The Henon Map

Suppose we devise some transformations.

Let us take a rectangle. Now, applying some transformation T' to it has the effect of pushing the middle up, into a horse-shoe shape, pointing upwards, leaving the vertical lines untouched. If another transformation T'' is then applied, which squashes the horse-shoe in. A final transformation T''' rotates the squashed horse-shoe so that it now points to the right.
A mathematical description of these transformations:

T': $x' = x$ and $y' = 1 + y - ax^2$
T'': $x'' = bx'$ and $y'' = y'$; where $-1 < b < 1$
T''': $x''' = y''$ and $y''' = x''$

Now, the full transformation, the composite of all above:

$$\begin{aligned} T &= T' \cdot T'' \cdot T''' \\ \Rightarrow x''' &= y'' = y' = 1 + y - ax^2 \\ y''' &= x'' = bx' = bx \end{aligned}$$

We can write this in 'map form', so that:

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned}$$

Which is the Henon map we had previously. Thus, we have given a geometrical 'derivation' of a map we know to posses chaotic behaviour. Thus, such a folding and stretching of initial conditions may induce chaotic behaviour.

7.3.1 Elementary Properties of the Henon Map

Invertible If trajectories are to be unique, then the process must be reversible. Each point has a unique path. To show that the Henon map is invertible, we want to solve for x_n, y_n in terms of x_{n+1}, y_{n+1}; i.e. to go backwards. Thus:

$$\begin{aligned} x_n &= b^{-1} y_{n+1} \\ y_n &= x_{n+1} - 1 + ax_n^2 \\ &= x_{n+1} - 1 + ab^{-2}(y_{n+1})^2 \end{aligned}$$

Therefore, the process is exactly invertible. Notice, not all maps will have this property, such as the Logistic.
Therefore, T^{-1} exists, for all $b \neq 0$.

The Henon map is dissipative If we look at an area, then apply T, the result should be smaller than the original.

Now, to decide if the 2D map:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned}$$

Is area preserving or otherwise, we must consider the Jacobian J of the system. Now (although we have not come across this yet) areas are transformed via:

$$dx'dy' = |\det J| dxdy$$

Thus, if $|\det J| < 1$, then the resultant area A' is smaller than the original area A. So, for the Henon map:

$$J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

And thus, $\det J = -b$, which full-fills $|\det J| < 1$, for all x, y, as we have the constraint on b that $-1 < b < 1 \Rightarrow |b| < 1$.
Therefore, the Henon map is 'area contracting'. Therefore, it posses an attractor of zero area.

Want some trapping region We have shown (or can show) that for certain parameter values, the Henon map has a trapping region.

Some trajectories result in escaping to ∞ Which is fairly obvious.

Therefore, we have constructed the Henon map from a basis constituting chaotic behaviour.

7.3.2 Attractors of the Henon Map

If we choose $b = 0.3, a = 1.4$, which is historically what Henon picked, we have negated the period-doubling zones, missed any other period-window, and are on the chaotic zone of the map. We find chaotic behaviour, with fractal patterns.

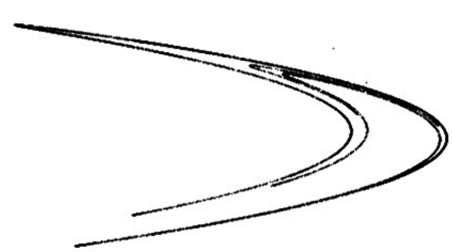


Figure 7: The Henon map. If any part of the graph is zoomed in on, it reveals more complexity. Hence, a fractal structure.