Optical Anisotropy:

$$\underline{D} = e_0 \underline{\underline{e}}\,\underline{E}$$

Suppose:

$$\underline{\underline{e}} = \begin{pmatrix} e_x & 0 & 0 \\ 0 & e_y & 0 \\ 0 & 0 & e_z \end{pmatrix}$$

Also, suppose that: $e_x = e_y$ $\qquad n = \sqrt{e}$

Notice that: $e_x = e_y = n_0^2$

$e_z = n_e^2$ thus a uniaxial crystal.

Now, $\nabla \cdot \underline{D} = 0$

$$\Rightarrow \qquad e_x \frac{\partial E_x}{\partial x} + e_y \frac{\partial E_y}{\partial y} + e_z \frac{\partial E_z}{\partial z} = 0$$

But, as $e_x \neq e_y \neq e_z$ (Generally. When this is the case, a biaxial crystal),

Then: $\nabla \cdot \underline{E} \neq 0$

Thus wave equation changes.
Hence, rederiving:

$$\nabla \times \underline{E} = -\dot{\underline{B}}$$

Taking curl:

$$\nabla \times \nabla \times \underline{E} = -\nabla \times \dot{\underline{B}} \qquad m = m_0 m_r$$

However: $\nabla \times \underline{B} = m e_0 \underline{\underline{e}}\,\underline{E}$

And, by identity: $\nabla \times \nabla \times \underline{E} = \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E}$

$$\Rightarrow \qquad \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E} = m e_0 \underline{\underline{e}}\,\ddot{\underline{E}}$$

Now, consider a plane-wave solution:

$$\underline{E} = \underline{\tilde{E}} \exp\{i(\underline{k} \cdot \underline{r} - wt)\}$$

So, evaluating the components to put into wave equation:

$$\nabla^2 \underline{E} = -k^2 \underline{E} \qquad \ddot{\underline{E}} = -w^2 \underline{E} \qquad \nabla \cdot \underline{E} = i\underline{k} \cdot \underline{E} \qquad \nabla(\nabla \cdot \underline{E}) = -\underline{k}(\underline{k} \cdot \underline{E})$$

Hence: $-\underline{k}(\underline{k}\cdot\underline{E})+k^{2}\underline{E}=w^{2}me_{0}\underline{\underline{e}}\,\underline{E}$

Thus: $-k^{2}\underline{E}+\underline{k}(\underline{k}\cdot\underline{E})=-w^{2}me_{0}\underline{\underline{e}}\,\underline{E}$

Now, look at the components:

$$-k^{2}E_{x}+k_{x}(\underline{k}\cdot\underline{E})=-w^{2}me_{0}e_{x}E_{x}$$
$$-k^{2}E_{y}+k_{y}(\underline{k}\cdot\underline{E})=-w^{2}me_{0}e_{y}E_{y}$$
$$-k^{2}E_{z}+k_{z}(\underline{k}\cdot\underline{E})=-w^{2}me_{0}e_{z}E_{z}$$

Once through the algebra, get:

$$\left(\frac{k_x^2}{n_0^2}+\frac{k_y^2}{n_0^2}+\frac{k_z^2}{n_0^2}-\frac{w^2}{c^2}\right)\left(\frac{k_x^2}{n_e^2}+\frac{k_y^2}{n_e^2}+\frac{k_z^2}{n_0^2}-\frac{w^2}{c^2}\right)=0$$

Assumed: $e_{x}=e_{y}=n_0^2 \quad e_{z}=n_e^2$

Which gives two different waves.

Now: $n\equiv\frac{c}{v_p} \qquad v_p\equiv\frac{w}{k} \qquad \Rightarrow n=\frac{ck}{w}\Rightarrow\frac{1}{n^2}=\frac{w^2}{c^2k^2}$

The first solution yields:

$$\frac{1}{n_0^2}=\frac{w^2}{c^2k^2}=\frac{1}{n^2} \qquad \Rightarrow \qquad n=\pm n_0$$

The “ordinary ray”

The second solution gives:

$$\frac{1}{n^2}=\frac{\sin^2 q}{n_e^2}+\frac{\cos^2 q}{n_e^2}$$

the “extraordinary ray”

$n(q)$