$$\sum_{n_\uparrow=0}^{N} \Omega(N, n_\uparrow) = \sum_{n_\uparrow} \frac{N!}{n_\uparrow!(N-n_\uparrow)!} = 2^N = \Omega(N)$$

$\Omega(N)$ = total number of microstates.
$\Omega(N, n_\uparrow)$ = total number of macrostates for a given microstate.

Boltzmann's law:

$$S = k_B \ln \Omega(N)$$

For large N, $\Omega(N,n) \approx$ Gaussian. Thus, fractional size of fluctuations from "mean" $\propto 1/\sqrt{N}$

Stirling's Approximation:

$$\boxed{\ln(n!) = n\ln n - n} \qquad + O(\ln n)$$

$$\text{as } n \to \infty$$

Now: $S = k_B \ln \Omega(N) = k_B \ln \sum_n \Omega(N,n)$

And the sum can be approximated by the biggest term in the series, i.e: $\Omega(N, N/2)$
Thus:

$$\ln \Omega(N) = \ln \Omega(N, N/2)$$

Statistical definitions:

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial E}\right)_{V,N} \qquad \frac{p}{T} \equiv \left(\frac{\partial S}{\partial V}\right)_{E,N} \qquad -\frac{m}{T} \equiv \left(\frac{\partial S}{\partial N}\right)_{E,V}$$

Boltzmann distribution:

Now:

$$S = k_B \ln \Omega = k_B \ln \Omega(E_0 - e_i)$$

For a system with E_0 total energy, in a state with energy e_i. Thus, can rearrange to:

$$\Omega(E_0 - e_i) = e^{\frac{1}{k_B} S(E_0 - e_i)}$$

Now, Taylor expanding the function in the exponential for $E_0 >> e_i$:

$$S(E_0 - e_i) = S(E_0) - e_i \left(\frac{\partial S}{\partial E}\right)_{V,N}\Bigg|_{E_0} + \frac{1}{2!} e_i^2 \left(\frac{\partial^2 S}{\partial E^2}\right)_{V,N}\Bigg|_{E_0} + \dots$$

Now:

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial E}\right)_{V,N}$$

$$\therefore \quad \left(\frac{\partial^2 S}{\partial E^2}\right)_{V,N} = \left(\frac{\partial}{\partial E}\left(\frac{1}{T}\right)\right)_{V,N} = -\frac{1}{T^2}\left(\frac{\partial T}{\partial E}\right)_{V,N}$$

$$= -\frac{1}{T^2 c_V}$$

Thus:

$$S(E_0 - e_i) = S(E_0) - \frac{e_i}{T} - \frac{e_i^2}{2T^2 c_V}$$

Or, if $c_V T >> e_i$:

$$S(E_0 - e_i) = S(E_0) - \frac{e_i}{T}$$

Now, going back to an original idea:

$$p_i(e_i) = \frac{\Omega(E_0 - e_i)}{\Omega(E_0)} \Rightarrow p_i \propto \Omega(E_0 - e_i)$$

Thus:

$$p_i = const.e^{\frac{1}{k_B}\left(S(E_0) - \frac{e_i}{T}\right)} = const.e^{-\frac{e_i}{k_B T}}$$

Let this constant be:

$\frac{1}{Z}$ which will be a normalisation constant

Thus:

$$p_i = \frac{1}{Z} e^{-\frac{e_i}{k_B T}} \qquad Z \equiv \sum_i e^{-\frac{e_i}{k_B T}}$$

Where p_i is the probability to find the system in a state with energy e_i. Z is basically a normalisation constant, but is known as the "partition function".

More accurately, if there is degeneracy $g(e_i)$ of a particular energy state e_i, then:

$$Z = \sum_i g(e_i) e^{-\frac{e_i}{k_B T}}$$

More usual to write, and is more convenient:

$$p_i = \frac{1}{Z} e^{-e_i b} \qquad Z = \sum_i g(e_i) e^{-e_i b} \qquad b \equiv \frac{1}{k_B T}$$

Now, the expectation value of the energy is:

$$\langle E \rangle = \sum_i p_i e_i = -\frac{1}{Z}\left(\frac{\partial Z}{\partial b}\right)_{N,V}$$
$$= k_B T\left(\frac{\partial \ln Z}{\partial T}\right)_{N,V}$$

And that for the heat capacity:

$$\langle c_V \rangle \equiv \left(\frac{\partial \langle E \rangle}{\partial T}\right)_{N,V} = k_B b^2 \left(\frac{\partial^2 \ln Z}{\partial b^2}\right)_{N,V}$$

It will be useful to know the hyperbolics relations:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{1 - e^{-2x}}{1 + e^{2x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$