

Derive the material derivative

Now, if:

$$\mathbf{r}(x, y, z, t) \quad (1.1)$$

Then:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial t} dt + \frac{\partial \mathbf{r}}{\partial x_i} dx_i \quad (1.2)$$

Divide (1.2) by  $dt$ , and take limit:

$$\lim_{dt \rightarrow 0} \left( \frac{d\mathbf{r}}{dt} \right) = \frac{\partial \mathbf{r}}{\partial t} + \lim_{dt \rightarrow 0} \left( \frac{dx_i}{dt} \right) \quad (1.3)$$

Now:

$$\lim_{dt \rightarrow 0} \left( \frac{dx_i}{dt} \right) = \frac{\partial x_i}{\partial t} = u_i \quad (1.4)$$

We define the LHS of (1.3) as the material derivative:

$$\lim_{dt \rightarrow 0} \left( \frac{d\mathbf{r}}{dt} \right) \equiv \frac{D\mathbf{r}}{Dt} \quad (1.5)$$

Hence, putting all together:

$$\frac{D\mathbf{r}}{Dt} = \frac{\partial \mathbf{r}}{\partial t} + u_i \frac{\partial \mathbf{r}}{\partial x_i} \quad (1.6)$$

Or, expanding out of suffix notation:

$$\boxed{\frac{D\mathbf{r}}{Dt} = \frac{\partial \mathbf{r}}{\partial t} + (\underline{u} \cdot \nabla) \mathbf{r}} \quad (1.7)$$

- the material derivative

Derive the integral form of the continuity equation

Now, the mass in volume  $V$ :

$$\int_V \rho dV \quad (2.1)$$

Rate of change of mass in  $V$ :

$$\frac{d}{dt} \int_V \mathbf{r} dV = \int_V \frac{\partial \mathbf{r}}{\partial t} dV \quad (2.2)$$

The rate of mass flow out of  $V$ , through a bounding closed surface  $S$ :

$$\int_S \mathbf{r} \underline{u} \cdot \underline{n} dS \quad (2.3)$$

Hence, if mass is conserved, (2.2) & (2.3) must balance:

$$\int_V \frac{\partial \mathbf{r}}{\partial t} dV = - \int_S \mathbf{r} \underline{u} \cdot \underline{n} dS \quad (2.4)$$

- the integral form of mass the continuity equation

Using divergence theorem & mass conservation in integral form, derive the pointwise form of mass conservation:

Divergence theorem:

$$\int_V (\nabla \cdot \underline{a}) dV = \int_S \underline{a} \cdot \underline{n} dS \quad (3.1)$$

Integral form of mass conservation:

$$\frac{d}{dt} \int_V \mathbf{r} dV = - \int_S \mathbf{r} \underline{u} \cdot \underline{n} dS \quad (3.2)$$

Now, the RHS of (3.2), using (3.1) becomes:

$$\int_S \mathbf{r} \underline{u} \cdot \underline{n} dS = \int_V (\nabla \cdot \mathbf{r} \underline{u}) dV \quad (3.3)$$

Now, expanding out the divergence in the RHS of (3.3), by vector calculus:

$$\nabla \cdot (\mathbf{r} \underline{u}) = \mathbf{r} (\nabla \cdot \underline{u}) + (\underline{u} \cdot \nabla) \mathbf{r} \quad (3.4)$$

Hence, (3.3) becomes:

$$\int_S \mathbf{r} \underline{u} \cdot \underline{n} dS = \int_V \{ (\underline{u} \cdot \nabla) \mathbf{r} + \mathbf{r} (\nabla \cdot \underline{u}) \} dV \quad (3.5)$$

Now, putting (3.5) into (3.2), and bringing over to the LHS:

$$\int_V \frac{\partial \mathbf{r}}{\partial t} dV + \int_V \{(\underline{u} \cdot \nabla) \mathbf{r} + \mathbf{r}(\nabla \cdot \underline{u})\} dV = 0 \quad (3.6)$$

Which can be brought under a single integral. The volume can be shrunk to a point. Hence:

$$\frac{\partial \mathbf{r}}{\partial t} + (\underline{u} \cdot \nabla) \mathbf{r} + \mathbf{r}(\nabla \cdot \underline{u}) = 0 \quad (3.7)$$

Which contains elements of the material derivative:

$$\frac{\partial \mathbf{r}}{\partial t} + (\underline{u} \cdot \nabla) \mathbf{r} \equiv \frac{D\mathbf{r}}{Dt} \quad (3.8)$$

Hence,

$$\frac{D\mathbf{r}}{Dt} + \mathbf{r}(\underline{u} \cdot \nabla) = 0 \quad (3.9)$$

- the pointwise form of the continuity equation

Derive equation for hydrostatic equilibrium

Now, suppose a body force  $\underline{F}(x,t)$  per unit mass, then the total body force is:

$$\int_V \mathbf{r} \underline{F} dV \quad (4.1)$$

Now, the total internal pressure force is:

$$- \int_S p \underline{n} dS \quad (4.2)$$

If the system is in equilibrium, (4.1) & (4.2) must balance. Hence:

$$\int_V \mathbf{r} \underline{F} dV = \int_S p \underline{n} dS \quad (4.3)$$

Now, if we use the divergence theorem in the form:

$$\int_V \nabla \cdot \mathbf{f} dV = \int_S \mathbf{f} \cdot \underline{n} dS \quad (4.4)$$

Thus, the RHS of (4.3) becomes:

$$\int_S p \underline{n} dS = \int_V \nabla p dV \quad (4.5)$$

Hence, (4.3) becomes:

$$\int_V (\mathbf{r}\underline{F} - \nabla p) dV = 0 \quad (4.6)$$

We can shrink the volume down to a point, hence, taking over to the other side:

$$\boxed{\mathbf{r}\underline{F} = \nabla p} \quad (4.7)$$

- the equation for hydrostatic equilibrium

### Derive Euler's equation

Now, if we start from Newton's 2<sup>nd</sup> law:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(m\underline{v}) = m \frac{d\underline{v}}{dt} + \underline{v} \frac{dm}{dt} \quad (5.1)$$

The analogue is:

$$\frac{d}{dt} \left\{ \int_V \underline{u} \mathbf{r} dV \right\} = - \int_S \mathbf{r} \underline{u} (\underline{u} \cdot \underline{n}) dS + \int_V \mathbf{r} \underline{F} - \nabla p dV \quad (5.2)$$

Or, rate of change of momentum is mass flux in plus the resultant force. Now, if looking at the middle integral in (5.2):

$$\int_S \mathbf{r} \underline{u} (\underline{u} \cdot \underline{n}) dS = \left( \int_S \mathbf{r} \underline{u} (\underline{u} \cdot \underline{n}) dS, \int_S \mathbf{r} \underline{v} (\underline{u} \cdot \underline{n}) dS, \dots \right) \quad (5.3)$$

Using the divergence theorem on one of the components of (5.3):

$$\int_S \mathbf{r} \underline{u} (\underline{u} \cdot \underline{n}) dS = \int_V \nabla \cdot (\mathbf{r} \underline{u} \underline{u}) dV \quad (5.4)$$

And, expanding out the RHS of (5.4):

$$\int_V \nabla \cdot (\mathbf{r} \underline{u} \underline{u}) dV = \int_V \{ \underline{u} \nabla \cdot (\mathbf{r} \underline{u}) + \mathbf{r} (\underline{u} \cdot \nabla) \underline{u} \} dV \quad (5.5)$$

Thus, doing this for all components of the RHS of (5.3), the LHS becomes:

$$\int_S \mathbf{r} \underline{u} (\underline{u} \cdot \underline{n}) dS = \int_V \{ \underline{u} (\nabla \cdot \mathbf{r} \underline{u}) + \mathbf{r} (\underline{u} \cdot \nabla) \underline{u} \} dV \quad (5.6)$$

Hence, the original equation, (5.2) becomes:

$$\int_V \frac{\partial}{\partial t}(\underline{r}\underline{u})dV + \int_V \{\underline{u}(\nabla \cdot \underline{r}\underline{u}) + \underline{r}(\underline{u} \cdot \nabla)\underline{u}\}dV - \int_V \{\underline{r}\underline{F} + \nabla p\}dV \quad (5.7)$$

Which can all be put under a single integral:

$$\int_V \left\{ \underline{r} \frac{\partial \underline{u}}{\partial t} + \underline{u} \frac{\partial \underline{r}}{\partial t} + \underline{u}(\nabla \cdot \underline{r}\underline{u}) + \underline{r}(\underline{u} \cdot \nabla)\underline{u} - \underline{r}\underline{F} + \nabla p \right\} dV = 0 \quad (5.8)$$

Now, notice that some of the components can be simplified:

$$\underline{r} \frac{\partial \underline{u}}{\partial t} + \underline{r}(\underline{u} \cdot \nabla)\underline{u} \equiv \underline{r} \frac{D\underline{u}}{Dt} \quad (5.9)$$

$$\underline{u} \left( \frac{\partial \underline{r}}{\partial t} + \nabla \cdot \underline{r}\underline{u} \right) = 0 \quad (5.10)$$

(5.10) by the continuity equation. Hence:

$$\int_V \left\{ \underline{r} \frac{D\underline{u}}{Dt} - \underline{r}\underline{F} + \nabla p \right\} dV = 0 \quad (5.11)$$

We can shrink the volume down to a point, and thus (5.11) becomes:

$$\underline{r} \frac{D\underline{u}}{Dt} - \underline{r}\underline{F} + \nabla p = 0 \quad (5.12)$$

Or:

$$\frac{D\underline{u}}{Dt} = \underline{F} - \frac{1}{\underline{r}} \nabla p \quad (5.13)$$

- Euler's equation

From Euler's equation, derive Bernoulli's equation

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\underline{r}} \nabla p + \underline{F} \quad (6.1)$$

Now:

$$\frac{1}{\underline{r}} \nabla p = \frac{1}{\underline{r}} \frac{\partial p}{\partial x_i} = \frac{d}{dp} \left[ \int \frac{dp}{\underline{r}} \right] \frac{\partial p}{\partial x_i} \quad \text{We can do this as } \underline{r} \text{ is "barotropic"}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial x_i} \int \frac{dp}{\mathbf{r}} \\
 &= \nabla \left( \int \frac{dp}{\mathbf{r}} \right)
 \end{aligned}$$

Hence:

$$\frac{1}{\mathbf{r}} \nabla p = \nabla \left( \int \frac{dp}{\mathbf{r}} \right) \quad (6.2)$$

Now, as  $\underline{F}$  is a “conservative force”, we can associate a scalar potential:

$$\underline{F} = -\nabla \Omega \quad (6.3)$$

Now, writing out the definition of the material derivative:

$$\frac{D\underline{u}}{Dt} \equiv \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \quad (6.4)$$

Now,  $\underline{u}$  is a “steady flow”. Thus:

$$\frac{\partial \underline{u}}{\partial t} = 0 \quad (6.5)$$

Now, there is a vector identity:

$$(\underline{u} \cdot \nabla) \underline{u} = \nabla \left( \frac{1}{2} \underline{u} \cdot \underline{u} \right) - \underline{u} \times \nabla \times \underline{u} \quad (6.6)$$

We shall define the vorticity vector,  $\underline{w} \equiv \nabla \times \underline{u}$ . Hence, putting (6.6) on the LHS of (6.1); and using (6.2) & (6.3) on the RHS... (5.1) becomes:

$$\nabla \left( \frac{1}{2} \underline{u} \cdot \underline{u} \right) - \underline{u} \times \underline{w} = -\nabla \left( \int \frac{dp}{\mathbf{r}} \right) - \nabla \Omega \quad (6.7)$$

Which can be brought under a common grad:

$$\nabla \left( \frac{1}{2} \underline{u} \cdot \underline{u} + \int \frac{dp}{\mathbf{r}} + \Omega \right) = \underline{u} \times \underline{w} \quad (6.8)$$

Now, if the flow is irrotational, i.e.  $\underline{w} = \underline{0}$ , then:

$$\boxed{\frac{1}{2} \underline{u} \cdot \underline{u} + \int \frac{dp}{\mathbf{r}} + \Omega = \text{const}} \quad (6.9)$$

- Bernoulli's equation

Prove:

$$\nabla(\mathbf{f}\underline{a}) = \mathbf{f}(\nabla \cdot \underline{a}) + (\underline{a} \cdot \nabla)\mathbf{f} \quad (\text{V1.1})$$

Now, in suffix notation, using the chain rule for differentiation:

$$\frac{\partial}{\partial x_i}(\mathbf{f}a_i) = \mathbf{f} \frac{\partial a_i}{\partial x_i} + a_i \frac{\partial \mathbf{f}}{\partial x_i} \quad (\text{V1.2})$$

And, putting back into vector notation:

$$\nabla \cdot (\mathbf{f}\underline{a}) = \mathbf{f}(\nabla \cdot \underline{a}) + (\underline{a} \cdot \nabla)\mathbf{f} \quad (\text{V1.3})$$

Use the divergence theorem to prove that:

$$\int_V \nabla \mathbf{f} dV = \int_S \mathbf{f} \underline{n} dS \quad (\text{V2.1})$$

Divergence theorem:

$$\int_V (\nabla \cdot \underline{a}) dV = \int_S \underline{a} \cdot \underline{n} dS \quad (\text{V2.2})$$

Now, let  $\underline{a} = \mathbf{f}\underline{c}$ , where  $\underline{c}$  an arbitrary constant vector. Thus:

$$\nabla \cdot \underline{a} = \nabla \cdot (\mathbf{f}\underline{c}) = \mathbf{f}(\nabla \cdot \underline{c}) + (\underline{c} \cdot \nabla)\mathbf{f} \quad (\text{V2.3})$$

Now, due to the definition of  $\underline{c}$ ,  $\nabla \cdot \underline{c} = 0$ . Thus:

$$\nabla \cdot \underline{a} = (\underline{c} \cdot \nabla)\mathbf{f} \quad (\text{V2.4})$$

Thus, (V2.2) becomes:

$$\int_V (\underline{c} \cdot \nabla)\mathbf{f} dV = \int_S \mathbf{f}\underline{c} \cdot \underline{n} dS \quad (\text{V2.5})$$

Hence:

$$\underline{c} \cdot \int_V \nabla \mathbf{f} dV = \underline{c} \cdot \int_S \mathbf{f} \underline{n} dS \quad (\text{V2.6})$$

Thus, we have proven:

$$\int_V \nabla \mathbf{f} dV = \int_S \mathbf{f} \underline{n} dS \quad (\text{V2.7})$$

Prove:

$$\nabla \times (\underline{f}\underline{a}) = \underline{f}(\nabla \times \underline{a}) + (\nabla \underline{f}) \times \underline{a} \quad (\text{V3.1})$$

Now, look at LHS:

$$\nabla \times \underline{f}\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underline{f}a_1 & \underline{f}a_2 & \underline{f}a_3 \end{vmatrix} \quad (\text{V3.2})$$

Now, looking only at the 'i' component:

$$\nabla \times \underline{f}\underline{a}|_i = \frac{\partial}{\partial y}(\underline{f}a_3) - \frac{\partial}{\partial z}(\underline{f}a_2) \quad (\text{V3.3})$$

Which can be expanded by the chain rule:

$$\begin{aligned} \nabla \times \underline{f}\underline{a}|_i &= \underline{f} \frac{\partial a_3}{\partial y} + a_3 \frac{\partial \underline{f}}{\partial y} - \underline{f} \frac{\partial a_2}{\partial z} - a_2 \frac{\partial \underline{f}}{\partial z} \\ &= \underline{f} \left( \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + \left( a_3 \frac{\partial \underline{f}}{\partial y} - a_2 \frac{\partial \underline{f}}{\partial z} \right) \\ &= \underline{f}(\nabla \times \underline{a})|_i + (\nabla \underline{f}) \times \underline{a}|_i \end{aligned}$$

Now, all other components will be similar. Hence:

$$\nabla \times (\underline{f}\underline{a}) = \underline{f}(\nabla \times \underline{a}) + (\nabla \underline{f}) \times \underline{a} \quad (\text{V3.4})$$

Prove:

$$\underline{u} \times \nabla \times \underline{u} = \nabla \left( \frac{1}{2} \underline{u} \cdot \underline{u} \right) - (\underline{u} \cdot \nabla) \underline{u} \quad (\text{V4.1})$$

Now, looking at part of the LHS, with  $\underline{u} = (u, v, w)$ :

$$\begin{aligned} \nabla \times \underline{u} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \underline{i} - \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) \underline{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{k} \end{aligned}$$

$$\underline{u} \times \nabla \times \underline{u}|_i = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u & v & w \\ \nabla \times \underline{u}|_i & \nabla \times \underline{u}|_j & \nabla \times \underline{u}|_k \end{vmatrix}$$



$$= v \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + w \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) \quad (\text{V4.2})$$

For convenience, define:

$$\begin{array}{lll} u_x = \frac{\partial u}{\partial x} & u_y = \frac{\partial u}{\partial y} & u_z = \frac{\partial u}{\partial z} \\ v_x = \frac{\partial v}{\partial x} & v_y = \frac{\partial v}{\partial y} & \dots \\ w_x = \frac{\partial w}{\partial x} & \dots & \dots \end{array}$$

Hence, (V4.2) becomes:

$$\begin{aligned} \underline{u} \times \nabla \times \underline{u} \Big|_i &= v v_x - v u_y + w w_x - w u_z \\ &= (u u_x + v v_x + w w_x) - (u u_x + v u_y + w u_z) \\ &= \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 \right) - (\underline{u} \cdot \nabla) \underline{u} \end{aligned}$$

Thus, putting together, and other components working similarly:

$$\underline{u} \times \nabla \times \underline{u} = \nabla \left( \frac{1}{2} \underline{u} \cdot \underline{u} \right) - (\underline{u} \cdot \nabla) \underline{u} \quad (\text{V4.3})$$