Cauchy’s Integral Formula:

$$f(z_0) = \frac{1}{2pi}\oint_g \frac{f(z)}{z - z_0}\,dz$$

g : Closed Jordan contour

$f(z)$ regular inside g

If g encloses a singularity:

$$\oint_g f(z)dz = 2pi$$

If $f(z)$ is regular inside g :

$$\oint_g f(z)dz = 0$$

An extension of the CIF:

$$f^{(n)}(z_0) = \left.\frac{d^{(n)}f}{dz^{(n)}}\right|_{z=z_0} = \frac{n!}{2pi}\oint_g \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

Where $n \geq 0$, and an element of the natural numbers.

Laurent Series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

Converges in the common region $R_1 \leq |z| \leq R_2$

Where:

$a_n z^n$ converges inside disc $|z| = R_2$

$b_n z^{-n}$ converges outside disc $|z| = R_1$

Thus, area of convergence is an annulus:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - a)^n$$ expansion about $z = a$

- Laurent Series

Laurent’s Theorem:

let $f(z)$ be regular in the annulus $R_1 \leq |z - z_0| \leq R_2$. Then, in the annulus:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

Where:

$$a_n = \frac{1}{2pi} \oint_g \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n = 0,1,2,...$$

$$b_n = \frac{1}{2pi} \oint_g (z - z_0)^{n-1} f(z) dz \qquad n = 1,2,3,...$$

For the *particular case only* of: $f(z)$ regular inside $|z - z_0| < R_1$
Then:

$$b_n = 0 \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

From the extension of the CIF.

Example of a Laurent expansion:

$$f(z) = \frac{1}{(1-z)(2-z)} \quad \text{about } z = 0$$

Separate by partial fractions:

$$f(z) = \frac{1}{1-z} - \frac{1}{2-z}$$ thus singularities at $z = 1$ & $z = 2$.

Now, suppose we expand $\frac{1}{1-z}$ in terms of $\frac{1}{z}$, then it will be valid outside $|z| > 1$

$\frac{1}{2-z}$ in terms of z, then valid inside $|z| < 2$

Just to see the sequence:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ... \qquad |z| < 1$$

Thus:

$$\frac{1}{2-z} = \frac{1}{2(1-\frac{z}{2})} = \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + ...\right] \qquad \text{on} \left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$$

Thus convergent inside $|z| < 2$.

Now, the other sequence:

$$\frac{1}{1-z} = \frac{1}{z(\frac{1}{z} - 1)} = -\frac{1}{z(1-\frac{1}{z})} = -\frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + ...\right] \text{on} \qquad \left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$$

Hence, adding:

$$f(z) = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - ... - \frac{z}{4} - \frac{z^8}{8} - ...$$

Valid on $1 < |z| < 2$

The Laurent expansion of *f(z)*.