Integral Transforms:

Laplace Transform:

$$\bar{f}(p) = L\{f(t)\} = \int_0^\infty f(t) e^{-pt} dt$$

$$L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\}$$ (linear operator)

Assume the infinity-limt has no contribution.

$$f(t) = e^{at} \qquad \bar{f}(p) = \frac{1}{p-a}$$

$$f(t) = t^n \qquad \bar{f}(p) = \frac{n!}{p^{n+1}}$$

$$L\left\{\frac{dy}{dt}\right\} = p\bar{y}(p) - y(0)$$

$$L\left\{\frac{d^2y}{dx^2}\right\} = p^2\bar{y}(p) - y'(0) - py(0)$$

e.g Solve:

$$\frac{dy}{dt} + y = e^{-2t} \qquad y(0) = 1$$

Take LT of whole equation:

$$p\bar{y}(p) - y(0) + \bar{y}(p) = \frac{1}{p+2}$$

$$(p+1)\bar{y} = \frac{1}{p+2} + 1 = \frac{p+3}{p+2}$$

$$\therefore \qquad \bar{y} = \frac{p+3}{(p+1)(p+2)}$$

Partial fractions:

$$\frac{p+3}{(p+1)(p+2)} = \frac{A}{p+1} + \frac{B}{p+2}$$

Thus:

$$p+3 = A(p+2) + B(p+1) \qquad [p = -1 \Rightarrow A = 2 \;\&\; p = -2 \Rightarrow B = -1]$$

$$\therefore \qquad \bar{y} = \frac{p+3}{(p+1)(p+2)} = \frac{2}{p+1} - \frac{1}{p+2}$$

Which can be inverted by inspection to give:

$$y(t) = 2e^{-t} - e^{-2t}$$

Bromwich Contour:

$$f(t) = L^{-1}\{\bar{f}(p)\} = \frac{1}{2pi}\int_{c-i\infty}^{c+i\infty} \bar{f}(p) e^{pt}\, dp$$

…an inversion integral. c chosen to the right of any singularities of $\bar{f}e^{pt}$. Now, if the conditions hold (it is directly analogous to Jordan's Lemma…):

- $\bar{f}$ has a finite number of isolated poles in the left half plane (LHP)
- $\left|\bar{f}\right| \to 0$ as $p \to \infty$ in LHP
- $m > 0$

then, only pick up contributions from the residues in LHP:

$$f(t) = \frac{1}{2pi}\left[2pi \sum_{z_k \text{ in LHP}} \operatorname{Re}s\{\bar{f}e^{pt}; z = z_k\}\right]$$

e.g solve:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^t \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

Hence:

$$p^2\bar{y} - y'(0) - py(0) + 3p\bar{y} + 3y(0) + 2\bar{y} = \frac{1}{p-1}$$

Collecting terms & inserting initial conditions:

$$(p^2 + 3p + 2)\bar{y} = \frac{1}{p-1} + 1 = \frac{p}{p-1}$$

$$\bar{y} = \frac{p}{(p-1)(p^2+3p+2)} = \frac{p}{(p-1)(p+2)(p+1)}$$

Inverting using the Bromwich contour…

$$y(t) = \frac{1}{2pi} \int_{c-i\infty} \underbrace{\frac{p}{(p-1)(p+2)(p+1)} e^{pt}}_{\equiv g(p)} dp$$

$$= \frac{1}{2pi} 2pi[\mathrm{Re}\, s\{g(p); p=1\} + \mathrm{Re}\, s\{g(p); p=-2\} + \mathrm{Re}\, s\{g(p); p=-1\}]$$

$$= \frac{1}{3 \cdot 2} e^{t} + \frac{-2}{-3 \cdot -2} e^{-2t} + \frac{-1}{-2 \cdot 1} e^{-t}$$

$$= \frac{1}{6} e^{t} - \frac{2}{3} e^{-2t} + \frac{1}{2} e^{-t}$$

Thus, the solution to the differential equation, subject to the boundary conditions, is:

$$y(t) = \frac{1}{6} e^{t} - \frac{2}{3} e^{-2t} + \frac{1}{2} e^{-t}$$

Other common transformations:

$$L\{e^{iwt}\} = \frac{p+iw}{p^2+w^2}$$

$$L\{\cos wt\} = \frac{p}{p^2+w^2}$$

$$L\{\sin wt\} = \frac{w}{p^2+w^2}$$

Convolution Theorem:

If we have $\bar{f}(p)\bar{g}(p)$, what is $L^{-1}\{\bar{f}(p)\dot{g}(p)\}$?
Now:

$$L\left\{\int_0^t f(s) g(t-s) ds\right\} = \bar{f}(p)\bar{g}(p)$$

$$\therefore \quad \boxed{L^{-1}\{\bar{f}(p)\bar{g}(p)\} = \int_0^t f(s) g(t-s) ds}$$

Fourier Transforms:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx \qquad \text{Transform}$$

$$f(x) = \frac{1}{2p} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk \qquad \text{Inverse}$$

e.g.

$$f(x)=e^{-|x|} \Rightarrow f(x)=\begin{cases} e^{-x} & x>0 \\ e^{x} & x<0 \end{cases}$$

thus:

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}dx \\ &= \int_{-\infty}^{0} e^{x}e^{ikx}dx+\int_{0}^{\infty} e^{-x}e^{ikx}dx \\ &= \int_{-\infty}^{0} e^{(1+ik)x}dx+\int_{0}^{\infty} e^{(ik-1)x}dx \\ &= \frac{1}{1+ik}+\frac{1}{1-ik} \\ &= \frac{2}{1+k^2} \end{aligned}$$

Thus, we can also write, via inversion:

$$e^{-|x|}=\frac{1}{p}\int_{-\infty}^{\infty}\frac{e^{-ikx}}{1+k^2}dk$$

Which highlights this method as being used to find the values of integrals.
Another example:

$$\begin{aligned} f(x) &= \begin{cases} 1-|x| & 0\le x<1 \\ 0 & |x|\ge 1 \end{cases} \\ &= \begin{cases} 1+x & -1\le x\le 0 \\ 1-x & 0<x<1 \\ 0 & elsewhere \end{cases} \end{aligned}$$

Making use of a pretty useful (derivable) result:

$$\int xe^{ikx}dx=\frac{x}{ik}e^{ikx}+\frac{1}{k^2}e^{ikx}$$

Thus, the Fourier transform of *f(x)* can be shown to be:

$$F(k)=\frac{2}{k^2}(1-\cos k)$$