Fourier Transforms: Derivation

… this is non-essential material, purely for interests sake!

Suppose we expand $f(x)$ on some range $-l/2 \le x \le l/2$ as a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(k_n x) + b_n \sin(k_n x)\}$$

Where

$$k_n \equiv \frac{2pn}{l}$$

$$a_0 = \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)dt \qquad a_n = \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)\cos(k_n t)dt$$

$$b_n = \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)\sin(k_n t)dt$$

Now, using the identity:

$$\cos\{k_n(t-x)\} = \cos(k_n t)\cos(k_n x) + \sin(k_n t)\sin(k_n x)$$

Giving:

$$f(x) = \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)dt + \frac{2}{l}\sum_{n=1}^{\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)\cos\{k_n(t-x)\}dt$$

Notice:

$$\cos\{k_n(t-x)\} = \frac{1}{2}\left[e^{ik_n(t-x)} + e^{-ik_n(t-x)}\right]$$

$$k_n \equiv \frac{2pn}{l} \Rightarrow k_{-n} = \frac{2p(-n)}{l} = -k_n \qquad \therefore \qquad k_{-n} = -k_n$$

Thus:

$$f(x) = \frac{1}{l}\sum_{n=-\infty}^{\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(t)e^{ik_n(t-x)}dt$$

Now, to tidy up, define:

$$h \equiv \frac{2p}{l} \qquad \Rightarrow \qquad k_n = nh$$

$$\therefore \qquad f(x) = \frac{1}{2p} \sum_{n=-\infty}^{\infty} h \int_{-\frac{p}{2}}^{\frac{p}{2}} f(t) e^{-nh(t-x)} dt$$

$$= \frac{1}{2p} \sum_{n=-\infty}^{\infty} hg(nh)$$

Where

$$g(nh) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} f(t) e^{inh(t-x)} dx$$

Now, as $h \to 0$:

$$\lim_{h \to 0} \sum_{-\infty}^{\infty} hg(nh) = \int_{-\infty}^{\infty} g(k) dk$$

Thus:

$$f(x) = \frac{1}{2p} \int_{-\infty}^{\infty} g(k) dk$$

$$= \frac{1}{2p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{ik(t-x)} dt dk$$

$$= \frac{1}{2p} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f(t) e^{ikt} dt dk$$

$$= \frac{1}{2p} \int_{-\infty}^{\infty} e^{-ikx} F(k) dk$$

Thus:

$$F(k) \equiv \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$ Fourier Transform

$$f(x) = \frac{1}{2p} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk$$ Inverse Fourier Transform