Complex Variables:

Some general relations:

$$z \;\; = z\bar{z} \qquad z^n = \left(re^{iq}\right)^n = r^n\left(\cos nq + i\sin nq\right)$$

$$e^{ip} + 1 = 0$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \cosh iz = \cos z$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \sinh iz = i\sin z$$

$$\ln z = \ln\left(re^{iq}\right) = \ln r + iq$$

… define Principle Value range to make this single valued

$$f(z) = u(x,y) + iv(x,y) \qquad \begin{matrix} \mathrm{Re}\{f(z)\} = u(x,y) \\ \mathrm{Im}\{f(z)\} = v(x,y) \end{matrix}$$

Continuity:

$$\left|f(z) - f(z_0)\right| \to 0 \qquad \text{as } z \to z_0 \text{ in any manner}$$

Differentiability:

$$\lim_{z \to z_0}\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \left.\frac{df}{dz}\right|_{z=z_0} \qquad \text{as } \left|z - z_0\right| \to 0 \text{ in any manner}$$

Cauchy-Riemann equations:

$f(z)$ regular if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{hold}$$

$u(x,y)$ & $v(x,y)$ are conjugate functions.
Both satisfy Laplace's equation:

$$\nabla^2 u = \nabla^2 v = 0$$

Integration by Parameterisation:

$$\int_g f(z)\,dz = \int_{t=a}^{t=b} f\{z(t)\}\frac{dz}{dt}\,dt$$

Defining a parameter *t* on *z*.

Cauchy's Theorem:

If $f(z)$ is regular inside a closed contour g, then:

$$\oint_g f(z)dz = 0$$

Cauchy's Integral Formula:

If $f(z)$ is regular in some domain D, and g some closed Jordan contour in D, and z_0 inside g, then:

$$f(z_0) = \frac{1}{2pi}\oint_g \frac{f(z)}{z - z_0}dz$$

Laurent series.

Need to know the following "common expansions":

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \qquad \text{convergent on } |z| < 1$$

$$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots \qquad \text{convergent on } |z| > 1$$

Now,

$\sum_n a_n z^n$ converges on some $|z| < R_2$

$\sum_n \frac{b_n}{z^n}$ converges on some $|z| > R_1$

Thus,

$\sum_n a_n z^n + \sum_n b_n z^{-n}$ converges on some $R_1 < |z| < R_2$

Generally:

$$f(z) = \sum_{-\infty}^{+\infty} A_n (z - z_0)$$ for a Laurent expansion about $z = z_0$

If, in Laurent expansion:

All	$b_n = 0$	$z = z_0$ removable singularity
For $n > k$	$b_n = 0$	pole of order k at $z = z_0$
If infinite number of b_n's		$z = z_0$ an essential singularity

$g(z)$ has a simple pole at $z = z_0$ if:

$$g(z_0) = 0 \qquad g'(z_0) \neq 0$$

$g(z)$ has a double pole at $z = z_0$ is:

$$g(z_0) = 0 \quad g'(z_0) = 0 \qquad g''(z_0) \neq 0$$

$g(z)$ has a pole order k at $z = z_0$ if:

$$f(z) = \frac{1}{g(z)} \qquad \text{has a zero of order } k \text{ at } z = z_0$$

Cauchy's Residue Theorem:

$f(z)$ regular inside a closed contour g , except at a finite number of isolated singularities. Then:

$$\oint_g f(z)dz = 2pi \sum_{z_k \text{ inside } g} \text{Res}\{f(z); z = z_k\}$$

Where:

$$\text{Res}\{f(z); z = z_0\} \equiv b_1$$

Which is the coefficient of the $\frac{1}{z - z_0}$ term in Laurent expansion.

Finding Residues:

Can also find residues by:

Let $$f(z) = \frac{f(z)}{(z - z_0)^m}$$

Where $f(z)$ has a pole order m at $z = z_0$ … $f(z_0) \neq 0$ … $f(z_0)$ regular

For

$m = 1$… simple pole: $\text{Res}\{f(z); z = z_0\} = f(z_0)$

$m = 2$… double pole: $\text{Res}\{f(z); z = z_0\} = \left.\frac{\partial f}{\partial z}\right|_{z=z_0}$

$m = m$… pole order m: $\text{Res}\{f(z); z = z_0\} = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} f}{\partial z^{m-1}}\right|_{z=z_0}$

another method:

$$f(z) = \frac{p(z)}{q(z)}$$ both $p(z)$ & $q(z)$ regular at $z = z_0$

$q(z_0)$ is a zero of order $m\ldots$ $p(z_0) \neq 0$

For simple poles:

$$\text{Res}\{f(z); z = z_0\} = \frac{p(z_0)}{q'(z_0)}$$

And, if $p(z_0) = 0$:

$$\text{Res}\{f(z); z = z_0\} = 2\frac{p'(z_0)}{q''(z_0)}$$

Provided $p'(z_0) \neq 0$, $q''(z_0) \neq 0$

Evaluation of real integral:

… use to solve integrals of the form:

$$\int_0^{2p} F(\cos q, \sin q)\, dq$$

Now, $z = re^{iq} = \cos q + i \sin q$, thus:

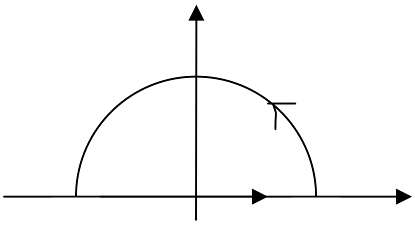
$$\cos q = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad dq = \frac{dz}{iz}$$

$$\sin q = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Thus, the real integral can be changed to a complex integral about the unit circle:

$$\oint_C G(z)\, dz$$ which can then be solved

Jordan's Lemma:



let g_R be the semi-circular contour, extending from +*R* to –*R*, and g_1 the portion of the real line joining these two points.
Now, let the closed contour be:

$$g = g_1 + g_R$$

Then, IF the following conditions hold for the integral:

$$\oint_g f(z)e^{imz}dz$$

$f(z)$ only has simple poles in the finite half of the upper-half-plane
$f(z) \to 0$ as $|z| \to \infty$
$m > 0$

Then, the integral becomes:

$$\oint_g f(z)e^{imz}dz = \int_{g_R} f(z)e^{imz}dz + \int_{g_1} f(z)e^{imz}dz$$

Jordan's Lemma says, that the contribution from $g_R \to 0$ as $R \to \infty$. Thus, the integral becomes just an integral along the real line:

$$\int_{-\infty}^{\infty} f(z)e^{imz}dz$$

Use for integrals of the form $\int_{-\infty}^{\infty} f(x)dx$