Differential Equations:

1st order ODE:

$$\frac{df}{dx} = -af \qquad a \text{ a constant}$$

Solution:

$$f(x) = Ae^{-ax}$$

2nd order ODE:

$$a\frac{d^2 f}{dx^2} + b\frac{df}{dx} + cf = 0$$

Solution:

$$f(x) = Ae^{l_1 x} + Be^{l_2 x}$$

Specific example:

$$\frac{d^2 f}{dx^2} = -k^2 f$$

Has solutions of the form:

$$f(x) = A\cos kx + B\sin kx$$

Wave Equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \qquad \text{PDE}$$

Solve by separating variables:

$$f(x,t) = X(x)T(t) \Rightarrow f = XT$$

Thus, equation becomes:

$$T\frac{d^2 X}{dx^2} = \frac{1}{c^2} X \frac{d^2 T}{dt^2}$$

Divide by $f = XT$:

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2}$$

The only way this is possible is if both sides are equal to a constant:

$$\frac{d^2X}{dx^2} = -k^2X \qquad \frac{d^2T}{dt^2} = -k^2c^2T$$

Which are just 2 ODE's.
The equation ***and*** boundary conditions specify an eigenvalue problem.

Normal Modes:

- Oscillations with definite frequency.

If X_1T_1 & X_2T_2 are solutions to $f = XT$, then so is:

$$f = X_1T_1 + X_2T_2$$

Thus:

$$f(x,t) = \sum_n X_n(x)T_n(t)$$ a linear superposition

For example, an eigenproblem under certain boundary conditions will give:

$$f(x,t) = \sum_{n=1}^{\infty} \sin\frac{npx}{L}\left(A_n \cos\frac{npct}{L} + B_n \sin\frac{npct}{L}\right)$$

Can find the coefficients A_n & B_n from Fourier series:

Fourier Series:

If:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{npx}{L} + b_n \sin\frac{npx}{L}\right)$$

on $-L \le x \le +L$, then:

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{npx}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{npx}{L}dx$$

Orthogonality:

$$\int_{-L}^{+L}\cos\frac{mpx}{L}\sin\frac{npx}{L}dx = 0$$

$$\int_{-L}^{+L} \cos\frac{npx}{L}\cos\frac{mpx}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

$$\int_{-L}^{+L} \sin\frac{npx}{L}\sin\frac{mpx}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

Due to symmetry:

Odd functions …. Sine functions
Even function …. Cosine function

Complex Series:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{inpx}{L}}$$

Two complex exponentials are orthogonal on $a \leq x \leq b$ if:

$$\int_a^b u*(x)v(x)dx = 0$$

Thus, on $-L \leq x \leq +L$:

$$\int_{-L}^{L}\left(e^{\frac{inpx}{L}}\right)*\left(e^{\frac{impx}{L}}\right)dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

Hence:

$$c_n = \frac{1}{2L}\int_{-L}^{+L} e^{-\frac{inpx}{L}} f(x)dx$$

Dirichlet's Conditions:

- $f(x)$ must be single valued and have a finite number of discontinuities;
- $\int_{-L}^{+L}|f(x)|dx$ must be finite

Hence, if these are satisfied, the its Fourier series converges to *f(x)*.
Eigenfunctions are orthogonal:

Supposed

$$\frac{d^2X_n}{dx^2} = -k_n^2 X_n \qquad \text{gives} \qquad X_n$$

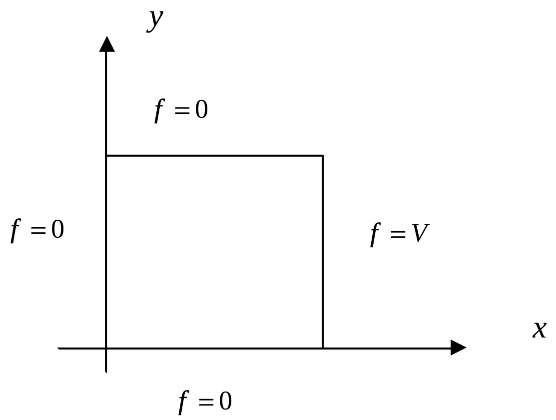
$$\frac{d^2X_m}{dx^2} = -k_m^2 X_m \qquad \qquad X_m$$

Then:

$$\int_0^L X_n X_m dx = 0 \qquad m \neq m$$

Example:

Laplace's equation: $\nabla^2 f = 0$



Square box with sides length L, in 2D.

Thus:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

With boundary conditions:

$$f(x,0) = f(x,L) = f(0,y) = 0$$
$$f(L,y) = V$$

Thus an eigenvalue problem.

So, start by separating variables: $f(x,y) = X(x)Y(y)$, resulting in:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Hence:

$$\frac{d^2X}{dx^2} = k^2 X \qquad \frac{d^2Y}{dy^2} = -k^2 Y$$

b.c's give:

$$Y_n(y) = \sin\frac{npy}{L}$$
$$X_n(x) = A_n e^{k_n x} + B_n e^{-k_n x}$$

Hence, solution is a linear superposition:

$$f(x,y)=\sum_{n=1}^{\infty}\left(A_n e^{\frac{npx}{L}}+B_n e^{-\frac{npx}{L}}\right)\sin\frac{npy}{L}$$

Impose more boundary conditions, gives:

$$A_n=-B_n \qquad \Rightarrow X_n=A_n\left(e^{k_n x}-e^{-k_n x}\right)=2A_n\sinh k_n x$$

And, after doing Fourier series:

$$f(x,y)=\frac{4V}{p}\sum_{\substack{n=1\\ odd}}^{\infty}\frac{1}{n\sinh(np)}\sinh\frac{npx}{L}\sin\frac{npy}{L}$$

Is the final solution.

Heat Flow Equation:

$$\nabla^2 T=\frac{1}{D}\frac{\partial D}{\partial t}$$

Where:

Thermal diffusivity: $D=\frac{k}{cr}$ $\qquad k=$ thermal conductivity

$c=$ specific heat capacity

Example:

b.c's: $\hat{n}\cdot\nabla f\big|_s=0$, $\left.\frac{\partial f}{\partial x}\right|_0=\left.\frac{\partial f}{\partial x}\right|_L=0$ $\qquad$ equation: $\frac{\partial^2 f}{\partial x^2}=\frac{1}{D}\frac{\partial f}{\partial t}$

Thus: $f(x,t)=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{npx}{L}e^{-g_n t}$

Where: $g_n=D\left(\frac{np}{L}\right)^2$ is the relaxation rate

Integral Transforms:

Fourier transform of $f(x)$:

$$g(k)=\frac{1}{2p}\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx$$

Inverse Fourier transform:

$$f(x)=\int_{-\infty}^{+\infty}g(k)e^{ikx}dx$$

Travelling Waves:

Phase velocity:

$$v_p = \left|\frac{w}{k}\right|$$

Group velocity:

$$v_g = \frac{dw}{dk}$$

If $v_p = c$, then non-dispersive wave.

Bandwidth:

For any function $f(x)$ and its transform $g(k)$, their widths satisfy:

$$\Delta x \Delta k \geq 1/2$$

Where $\Delta x, \Delta k$ are the widths of the function and the transform.

Convolution:

$$F(x) = \int_{-\infty}^{+\infty} f_1(x - x') f_2(x') dx'$$

- The convolution of two functions.

f_2 is a "smearing" function

e.g. in astronomy: $f_1 \rightarrow$ true image

$f_2 \rightarrow$ effects of telescope

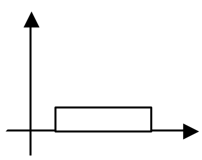
- symmetric in f_1 and f_2

Fourier transforms of the convolution of two functions is the product of their transforms:

$$G(k) = 2p g_1(k) g_2(k)$$

Dirac *d* -Function:

- an infinitely narrow, sharp spike



Suppose a uniform source, centred on x_0, with width $2a$

$$f(x) = \begin{cases} 1/2a & x_0 - a < x < x_0 + a \\ 0 & elsewhere \end{cases}$$

Let $a \to 0$, but keep area fixed at 1.
In the limit $a \to 0$:

$$f(x) = d(x - x_0)$$

Which has the properties:

$$d(x) = 0 \qquad x \neq 0$$

$$\int_{-\infty}^{+\infty} d(x)\,dx = 1$$

Series Expansions:

Taylor, around the point x_0:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} a_n (x - x_0)^2$$

With:

$$a_n = \frac{1}{n!} f^n(x_0)$$

Convergence:

If: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} x \right| < 1$ then the series converges

Also works for: $\lim_{n\to\infty} \left| \frac{a_{n+2}}{a_n} x^2 \right| < 1$

Linear Independence:

$$\sum_{n=0}^{\infty} a_n x^n = 0$$

Example:

Show that: $y = a_0 + a_2 x^2$ is a solution to:

$$\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

Now:

$$\frac{dy}{dx} = 2a_2 x$$

$$\frac{d^2 y}{dx^2} = 2a_2$$

Sub into equation:

$$2a_2 - 4x^2 a_2 + 2n\left(a_0 + a_2 x^2\right) = 0$$

$$\Rightarrow 2\left(a_2 + na_0\right) + 2(n-2)a_2 x^2 = 0$$

By linear independence:

$$a_2 + na_0 = 0$$

$$n - 2 = 0 \Rightarrow n = 2$$

$$\Rightarrow a_2 = -2a_0$$

Hence:

$$y = a_0(1 - 2x) \quad n = 2$$

Legendre Polynomials:

Legendre's equation:

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell + 1)y = 0 \qquad -1 \le x \le +1$$

Can show that:

$$a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n-1)} a_n$$

A recurrence relationship

Thus, all coefficients can be written in terms of a_0 & a_1.

Hence:

If	$a_0 = 0$	all even terms = 0
If	$a_1 = 0$	all odd terms = 0

Polynomial terminates at $n = \ell$. Here are a few polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}\left(3x^2 - 1\right)$$

Legendre's equation is an eigenvalue problem:

Eigenfunctions: $P_\ell(x)$ Eigenvalues: $\ell(\ell+1)$

Boundary condition: finite at ± 1

Hence:

$$\int_{-1}^{=1} P_\ell(x) P_m(x)\,dx = \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{2\ell+1} & \ell = m \end{cases}$$

Thus orthogonal

So, can produce a Legendre series:

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

Where:

$$c_\ell = \frac{2\ell+1}{2}\int_{-1}^{+1} f(x) P_\ell(x)\,dx$$

Bessel Functions:

Bessel's equation:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - m^2\right)y = 0$$

Boundary condition: finite at $x = 0$.
Solution of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s} \qquad s \text{ a number}$$

$$a_n = -\frac{1}{(n+s)^2 - m^2} a_{n-2}$$

If $s = +m \Rightarrow J_m(x)$ Regular at $x = 0$.
Regular Bessel functions.

If $s = -m \Rightarrow N_m(x)$ Singular at $x = 0$.
Irregular Bessel functions.

2- and 3-D Problems:

Rectangular membrane: x-length = a, y-length = b.

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Sides fixed… thus b.c's: $f(0,y,t) = f(a,y,t) = f(x,0,t) = f(x,b,t) = 0$
Separable solutions.

Gives:

$$k_{n_x} = \frac{n_x p}{a} \qquad n_x = 1,2,3,...$$

$$X_{n_x}(x) = \sin k_{n_x} x = \sin \frac{n_x p x}{a}$$

$$k_{n_y} = \frac{n_y p}{b} \qquad n_y = 1,2,3,...$$

$$Y_{n_y}(x) = \sin k_{n_y} y = \sin \frac{n_y p y}{b}$$

$$T(t) = A\cos wt + B\sin wt$$

$$w^2 = c^2\left(k_x^2 + k_y^2\right)$$

Thus, the solution is:

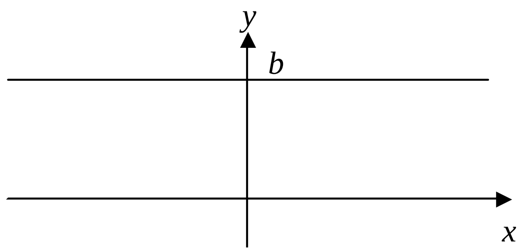
$$f_{n_x n_y}(x,y,t) = X_{n_x} Y_{n_y}\left(A_{n_x n_y}\cos\left(w_{n_x n_y} t\right) + B\sin\left(w_{n_x n_y} t\right)\right)$$

Where:

$$w_{n_x n_y} = c\sqrt{k_x^2 + k_y^2} = c\sqrt{\frac{n_x^2 p^2}{a^2} + \frac{n_y^2 p^2}{b^2}}$$

Thus, get degeneracies and nodal lines.

Waveguide:



Again: $\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$

But the boundary conditions are:

$$f(x,0,t) = f(x,b,t) = 0$$

Gives:

$$f(x,y,t) = A\sin \frac{npy}{b} e^{i(k_x x - wt)} \qquad k_y = \frac{np}{b}$$

$$n = 1,2,3,... \qquad k_x = \text{anything!}$$

Travelling wave:

$$w = c\sqrt{k_x^2 + \frac{n^2 p^2}{b^2}} \qquad \text{a dispersion relation}$$

$$v_g = \frac{dw}{dk_x}$$

Cut-off frequency for n-th mode:

$$w_{nc} = c\frac{np}{b}$$

Thus, if $w < w_{nc}$, then get evanescent waves… attenuated waves.

Hot Plate:

A circular disc on the *x*,*y*-plane.
Boundary conditions:

$$\left.\frac{\partial f}{\partial r}\right|_{r=a} = 0 \qquad \text{thus "insulated edges"}$$

$$\Phi(f) = \Phi(f + 2p) \qquad \text{gives periodicity of the angular coordinate.}$$

Gives a solution:

$$f(r, f, t) = J_m(kr)\left[A_m \cos mf + B_m \sin mf\right]e^{-gt}$$

$$g = Dk^2$$

Waves on a Sphere:

Use ∇^2 in spherical polars.
The boundary conditions:

$$f(q, f, t) = f(q, f + 2p, t)$$

Regular at $q = 0, p$