# Separation of Variables

The equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Has solutions of the form:

$$f(x,t) = X(x)T(t)$$

So, evaluating the derivatives:

$$\frac{\partial^2 f}{\partial x^2} = T\frac{d^2 X}{dx^2} \qquad \frac{\partial^2 f}{\partial t^2} = X\frac{d^2 T}{dt^2}$$

Hence, putting back into PDE:

$$T\frac{d^2 X}{dx^2} = \frac{1}{c^2} X \frac{d^2 T}{dt^2}$$

Diving by $f = XT$ :

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

These can only be equal if:

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k^2 = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

Hence giving two normal differential equations. Which can be solved easily.
Now, if $X_1(x)T_1(t)$ & $X_2(x)T_2(t)$ are solutions, then by superposition, so is:

$$f = X_1(x)T_1(t) + X_2(x)T_2(t)$$

Infact, so is:

$$f(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$$

For a particular set of boundary conditions, a solution is:

$$f(x,t) = \sum_{n=1}^{\infty} \sin\frac{npx}{L}\left(A_n \cos\frac{npct}{L} + B_n \sin\frac{npct}{L}\right)$$

And the coefficients $A_n$ & $B_n$ can be found via Fourier series.

The solution $X_n(x)$ is an eigenfunction, with eigenvalue $k_n^2$. This is because it conforms to the eigenproblem:

$$\frac{d^2}{dx^2} X_n = -k_n^2 X_n$$

When coupled with boundary conditions.
A useful relationship, to tidy expressions up, is:

$$\cos(np) = (-1)^n \qquad n = 1,2,3,...$$

Now, two eigenfunctions & eigenvalues for a problem satisfy:

$$\frac{d^2}{dx^2} X_n = -k_n^2 X_n$$
$$\frac{d^2}{dx^2} X_m = -k_m^2 X_m$$

Now, it can be shown that:

$$\int_0^L X_n X_m dx = 0 \qquad m \neq n$$

Which implies that the eigenfunctions $X_n$ & $X_m$ are orthogonal. Hence, eigenfunctions are orthogonal.

## Laplace's Equation:

$$\nabla^2 f = 0$$

Under the boundary conditions:

- $f(x,0) = f(x,L) = f(0,y) = 0$
- $f(L,y) = V$

Which is a square box, with all but one side at potential 0, and the other at *V*.
The solutions are of the form:

$$f(x,y) = \sum_{n=1}^{\infty} \left( A_n e^{\frac{npx}{L}} + B_n e^{-\frac{npx}{L}} \right) \sin \frac{npy}{L}$$

And use Fourier series (for complex exponentials) to find the coefficients.

## The Heat Flow Equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{D}\frac{\partial f}{\partial t} \qquad \text{where } D = \frac{k}{Cr} \qquad \text{- thermal diffusivity}$$

$k$ = thermal conductivity

And $f(x,t)$ gives the temperature at $(x,t)$.
The boundary conditions:

$$\left.\frac{\partial f}{\partial x}\right|_0 = \left.\frac{\partial f}{\partial x}\right|_L = 0$$

Over a length *L*, give a solution of the form:

$$f(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{npx}{L} e^{-g_n t}$$

Where: $g_n = D\left(\frac{np}{L}\right)^2$ "relaxation rate"

Hence can find coefficients from cosine Fourier series.