Lagrangian:

$\underline{v} = \dot{r}\underline{\hat{r}} + r\dot{q}\underline{\hat{q}}$ in polars

Small amplitude oscillations: $q \ll 1$

$$\sin q \sim q \qquad \cos q \sim 1 - \frac{q^2}{2}$$

Solutions to the SHM equations: $q_i = A_i e^{iwt}$

Torque: $\underline{t} = \underline{r} \times \underline{F}$

Angular momentum: $\underline{L} = \underline{r} \times \underline{p}$ $\quad \frac{d\underline{L}}{dt} = \underline{t}$

Kinetic energy: $T = T_{trans} + T_{rot}$

$$T_{trans} = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2$$

$$T_{rot} = \frac{1}{2}I\Omega^2 \qquad v = r\Omega \Rightarrow \Omega = \frac{v}{r}$$

Thus: $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\frac{\dot{r}^2}{r^2} = \frac{1}{2}\dot{r}^2\left(m + \frac{I}{r^2}\right)$

The rotation creates a "mass correction" term.

D'Alembert's principle:

System of 'n'-particles, with 'S'-constrains, hence, the number of degrees of freedom 'N' is:

$$N = 3n - S$$

Get N generalised coordinates.

Lagrange Equation: $L(q_i, \dot{q}_i, t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad L = T - U$$

Conservation of angular momentum: $\frac{d}{dt}\left(mr^2\dot{q}\right) = 0$

Properties:

Homogeneity of time:

Lagrangian of a closed system does not depend upon time:

$$E = T + V = const$$

Homogeneity of time → Conservation of energy.

Homogeneity of space:

Lagrangian of a closed system remains unchanged under parallel displacement of the whole system in space.

$$\underline{p} = \sum_i m_i \underline{v}_i = const$$

Homogeneity of space → Conservation of linear momentum.

Isotropy of space:

Lagrangian of a closed system is invariant under rotation transformations.

$$\sum_i \underline{r}_i \times \underline{p}_i = const$$

Isotropy of space → Conservation of angular momentum.

If symmetry shows that rotation about *z*-axis invariant, then L_z is conserves.

Noether's theorem: Symmetry of system → Respective conservation law.

Calculus of Variations:

A path length between two points is given by, in Cartesian space:

$$S = \int_A^B ds$$

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (y')^2}$$

Hence:

$$S = \int_A^B \sqrt{1 + (y')^2}\, dx \qquad y' = \frac{dy}{dx}$$

Thus: $S = \int_A^B f(x, y, y')dx$

$$I\{y(x)\} = \int_{x_1}^{x_2} f(x, y, y')dx \qquad y' = \frac{dy}{dx}$$

And, Euler's equation can be used to find the extremum:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Now, for the Cartesian $f(x, y, y') = \sqrt{1 + (y')^2}$:

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{1}{2}\left(1 + (y')^2\right)^{\frac{1}{2}} 2y' = \frac{y'}{\sqrt{1 + (y')^2}}$$

Hence:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} = const = a$$

Hence: $y' = \dfrac{dy}{dx} = const = a \quad \Rightarrow \quad y = ax + b$

A straight line.

Principle of Least Action

$$A = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt$$ - like Euler's equation, but with the Lagrangian.

Evaluate the integral over a whole period: $2p/w$.
To minimise the action, with respect to the amplitude of a vibration c_i:

$$\frac{\partial A}{\partial c_i} = 0$$

Hamiltonian:

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L$$

Generalised momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If $\frac{\partial H}{\partial t} = 0$, then $H = E$… conservation of energy