

## Maxwell's Equations:

Displacement current:

$$i_D = \epsilon_0 \frac{\partial \mathbf{f}}{\partial t}$$

Thus, Ampere's law becomes Maxwell's 3<sup>rd</sup> equation:

$$\nabla \times \underline{B} = \underline{j} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

Hence, all of Maxwell's equations:

$$\begin{array}{ll} \nabla \cdot \underline{E} = \frac{\underline{r}}{\epsilon_0} & \nabla \cdot \underline{B} = 0 \\ \nabla \times \underline{B} = \underline{j} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} & \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \end{array}$$

Or, in an alternative form:

$$\begin{array}{l} \nabla \cdot \underline{D} = \underline{r}_f \\ \nabla \times \underline{H} = \underline{j}_f + \frac{\partial \underline{D}}{\partial t} \end{array}$$

## EM-Waves:

Maxwell's equations in free space read:

$$\begin{array}{ll} \nabla \cdot \underline{E} = 0 & \nabla \cdot \underline{B} = 0 \\ \nabla \times \underline{B} = \epsilon_0 \frac{\partial \underline{E}}{\partial t} & \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \end{array}$$

For the z-direction, say... they reduce to:

$$\frac{\partial E}{\partial z} = -\frac{\partial E}{\partial t} \quad \frac{\partial B}{\partial z} = -\epsilon_0 \frac{\partial E}{\partial t}$$

Thus, get a 1D wave equation:

$$\frac{\partial^2 E}{\partial z^2} = \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \frac{\partial^2 B}{\partial z^2} = \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Or, in 3D:

$$\nabla^2 \underline{E} = \underline{m}_0 \underline{e}_0 \frac{\partial^2 E}{\partial t^2}$$

And:

$$c^2 = \frac{1}{\underline{m}_0 \underline{e}_0} \quad \text{which is the speed of EMR in vacuum}$$

Solutions to wave equations have the form:

$$f(z - vt)$$

e.g:

$$\underline{E} = \underline{E}_0 e^{i(kz - \omega t)}$$

Relation between  $\underline{E}$  and  $\underline{B}$ :

$$\underline{B} = \frac{1}{c} \hat{k} \times \underline{E} \quad \Rightarrow \quad B_y = \frac{1}{c} E_x$$

Where  $\hat{k}$  is a unit vector in the direction of propagation.

Energy density in the wave is given by:

$$U = \frac{1}{2} \int (\underline{E} \cdot \underline{D} + \underline{B} \cdot \underline{H}) dt$$

$$\begin{aligned} \text{Or: } U &= \frac{1}{2} \left( \underline{e}_0 E^2 + \frac{1}{\underline{m}_0} B^2 \right) \\ &= \frac{1}{2} \left( \underline{e}_0 E_{0x}^2 + \frac{1}{\underline{m}_0} B_{0y}^2 \right) \cos^2(kz - \omega t) \end{aligned}$$

$$\begin{aligned} E_{0x} &= c B_{0y} \\ &= \frac{1}{2} \left( \underline{e}_0 E_{0x}^2 + \frac{1}{\underline{m}_0 c^2} E_{0x}^2 \right) \cos^2(kz - \omega t) \\ &= \underline{e}_0 E_{0x}^2 \cos^2(kz - \omega t) \end{aligned}$$

Thus, electric and magnetic fields carry the same amount of energy.

**Poynting Vector:**

$$\underline{N} \quad \text{Wm}^{-2}$$

Energy flux – energy crossing unit area per second.

$$N = cU (= cE_0^2 \mathbf{e}_0 \cos^2(kz - \omega t))$$

Thus:

$$N = \frac{1}{\mathbf{m}_0} E_x B_y (= E_x H_y)$$

$$\underline{N} = \frac{1}{\mathbf{m}_0} \underline{E} \times \underline{B} \quad (= \underline{E} \times \underline{H})$$

e.g.:

$$N = \frac{1}{\mathbf{m}_0 c} E_0^2$$

Impedance of free space:

$$= \left( \sqrt{\frac{\mathbf{e}_0}{\mathbf{m}_0}} \right)^{-1} = 377 \Omega$$

Irradiance:

- a time averaged Poynting vector

$$\langle \underline{N} \rangle = \frac{1}{2\mathbf{m}_0} \frac{E_0^2}{c}$$

Radiation Pressure:

$$\begin{array}{l} \text{Absorbed:} \quad \underline{P} = \frac{\langle \underline{N} \rangle}{c} \\ \text{Reflected:} \quad \underline{P} = \frac{2\langle \underline{N} \rangle}{c} \end{array}$$

2 from conservation of momentum

Force exerted:  $F = PA$  pressure times area

Reflection from a conductor: No  $\underline{E}$ -field inside... thus, reflected and incident wave must cancel at boundary... thus change of sign and direction of wave.

$$E_T + E_R = 0 \text{ at boundary.}$$

Polarisation:

Plane/Linear Polarisation:

Electric field confined to a single plane

If the phase difference between components is  $\mathbf{p}$ , and the amplitudes are the same, then they are in phase.

Circular/Elliptical Polarisation:

$$\hat{x}E_{0_x} \cos(kz - \omega t) + \hat{y}E_{0_y} \cos(kz - \omega t + \mathbf{d})$$

Any phase difference other than  $\mathbf{p}$  will produce elliptical polarisation, with a phase difference of  $\mathbf{p} / 2$  giving circular polarisation:

Right Circular:

$$\hat{x}E_{0_x} \cos(kz - \omega t) + \hat{y}E_{0_y} \sin(kz - \omega t)$$

Left Circular:

$$\hat{x}E_{0_x} \cos(kz - \omega t) - \hat{y}E_{0_y} \sin(kz - \omega t)$$

A phase shifter – or retarder – will add phase differences.

EM Waves in Materials:

$$\begin{aligned} \nabla \cdot \underline{D} &= \mathbf{r}_f & \nabla \cdot \underline{B} &= 0 \\ \nabla \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} & \nabla \times \underline{H} &= \underline{j} + \frac{\partial \underline{D}}{\partial t} \end{aligned}$$

Dielectrics:

$$\begin{aligned} \mathbf{e}_0 &\rightarrow \mathbf{e}\mathbf{e}_0 \\ \mathbf{m}_0 &\rightarrow \mathbf{m}\mathbf{m}_0 \end{aligned}$$

Assume no charges/currents. Thus, the wave equations become:

$$\nabla^2 \underline{E} = \mathbf{m}\mathbf{m}_0\mathbf{e}\mathbf{e}_0 \frac{\partial^2 \underline{E}}{\partial t^2} \quad \text{Similarly for } \underline{B}.$$

Now:

$$v = \frac{1}{\sqrt{\mathbf{m}\mathbf{m}_0\mathbf{e}\mathbf{e}_0}} = \frac{c}{\sqrt{\mathbf{m}\mathbf{e}}}$$

But:  $v = \frac{c}{n}$

Thus:

$$n = \sqrt{\mathbf{m}\mathbf{e}} \sim \sqrt{\mathbf{e}}$$

As  $\mathbf{m} \sim 1$  for most dielectrics.

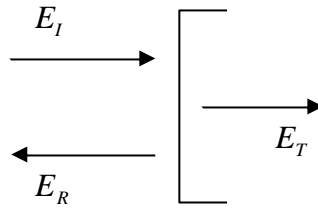
Snell's Law:

$$n_1 \sin \mathbf{q}_1 = n_2 \sin \mathbf{q}_2$$

$$k_I \sin \mathbf{q}_I = k_R \sin \mathbf{q}_R = k_T \sin \mathbf{q}_T$$

- generally

From boundary conditions.



Going from vacuum into a dielectric

Boundary conditions say:

$$E_I + E_R = E_T$$

$$H_I + H_R = H_T$$

Thus:

$$\frac{B_I}{\mathbf{m}_0} - \frac{B_R}{\mathbf{m}_0} = \frac{B_T}{\mathbf{m}\mathbf{m}_0}$$

And:  $B = \frac{1}{c} E = \sqrt{\mathbf{e}_0 \mathbf{m}_0} E$

Hence:

$$E_I \sqrt{\frac{\mathbf{e}_0}{\mathbf{m}_0}} - E_R \sqrt{\frac{\mathbf{e}_0}{\mathbf{m}_0}} = E_T \sqrt{\frac{\mathbf{e}\mathbf{e}_0}{\mathbf{m}\mathbf{m}_0}}$$

Using the approximation above, and putting in the refractive index:

$$E_I - E_R = n E_T$$

Hence, the reflection and transmission coefficients are:

$$R = \frac{E_R^2}{E_I^2} = \left( \frac{1-n}{1+n} \right)^2 \qquad T = \frac{E_T^2}{E_I^2} \frac{v}{c} = \frac{4n}{(1+n)^2}$$

Note that:  $T + R = 1$

Conducting Medium:

$$\underline{j} = \mathbf{s} \underline{E}$$

“Ohm’s Law”

Hence, Maxwell 3 becomes:

$$\nabla \times \underline{B} = \underline{m_0 s} \underline{E} + \underline{e_0 m_0} \frac{\partial^2 \underline{E}}{\partial t^2}$$

In a good conductor:

$$\underline{m_0 j} \gg \underline{e_0 m_0} \frac{\partial^2 \underline{E}}{\partial t^2}$$

Thus, the equation to work with is:

$$\nabla \times \underline{B} = \underline{m_0 s} \underline{E}$$

Skin Depth:

$$d = \sqrt{\frac{2}{\underline{m_0 s w}}} \quad \text{an attenuation factor.}$$

Thus, the resistance becomes:

$$R = R_0 \frac{r}{2d}$$

Plasmas:

Using:  $\underline{F} = m\dot{\underline{r}}$        $\underline{F} = q\underline{E}$        $\underline{j} = Ne\dot{\underline{r}}$

Can get:

$$\nabla^2 \underline{E} = \underline{m_0 n_e} e \left( \frac{e\underline{E}}{m_e} \right) + \underline{e_0 m_0} \frac{\partial^2 \underline{E}}{\partial t^2}$$

Below the "plasma frequency":

$$\omega_p = \sqrt{\frac{n_e e^2}{\underline{e_0 m_e}}}$$

Incident electromagnetic waves are strongly attenuated.  
Derived from finding the complex  $k$ .

*Phase Velocity:*

$$v_p = \frac{\omega}{k}$$

*Group Velocity:*

$$v_g = \frac{d\omega}{dk}$$