

No damping, no force:

$$\ddot{x} = -\omega^2 x$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$x = A \sin(\omega t + \phi) = A e^{i(\omega t + \phi)}$$

Damping force b . no forcing term:

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solution: $x = A e^{i\omega - \frac{g}{2}t}$ $g = \frac{b}{m}$

$$\Rightarrow \omega = \omega_0 - \frac{g^2}{4} \quad (\omega_0 = \sqrt{\frac{k}{m}})$$

So, the damping term is $e^{-\frac{gt}{2}} = e^{-\frac{bt}{2m}}$, so bigger $b \Rightarrow$ faster damping

$$Q = \frac{\omega_0}{g} \quad \text{Quality factor of the motion}$$

So: $\omega = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$

Forced & damped:

$$m\ddot{x} + b\dot{x} + kx = F e^{i\omega t}$$

As before: $\omega_0 = \sqrt{\frac{k}{m}} \quad \omega = \omega_0 - \frac{g^2}{4} \quad g = \frac{b}{m} \quad Q = \frac{\omega_0}{g}$

Solution: $x = A' e^{i\omega t}$

Hence, the equation of motion can be written as:

$$-m\omega^2 A' + i\omega b A' + k A' = F$$

Where $A' = \frac{F}{-m\omega^2 + k + i\omega b}$ - which is a 'response' amplitude/phase

Alternatively: $A' = \frac{F/m}{\omega_0^2 - \omega^2 + i g \omega}$

Taking the real part:

$$A = \frac{F/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + g^2 \omega^2}} \quad (\text{amplitude})$$

It appears that: $A' = A e^{i\mathbf{d}}$, with $\tan \mathbf{d} = \frac{g\omega}{\omega_0^2 - \omega^2}$ (phase)

$$\omega_{\max} = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

LC-Circuits:

Inductor	L	$Z = j\omega L$	$V = L \frac{dI}{dt} = LI\dot{}$	
Charge	Q			
Resistance	R	$Z = R$	$V = IR$	($Z = \text{impedance}$)
Voltage	V			
Capacitor	C	$Z = 1/j\omega C$	$V = Q/C$	
Voltage across capacitor:		$V_c = \frac{Q}{C}$		
Voltage across inductor;		$V_L = LI\dot{}$		

So, for (just) an LC circuit:

$$LI\dot{} + \frac{Q}{C} = 0 \qquad \dot{Q} = I$$

⇒

$$L\ddot{I} + \frac{1}{C}I = 0 \qquad \omega = \frac{1}{\sqrt{LC}}$$

So, the solution is: $I = I_0 \sin(\omega t + \phi)$

Add a resistor R (like the damping term before):

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$

→ with: $\omega_0 = \frac{1}{\sqrt{LC}}$ and $g = \frac{R}{L}$

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = 0$$

So, the quality term: $Q = \frac{1}{R} \sqrt{\frac{L}{C}}$ (Q is NOT the charge above!!!)

Add a voltage source $V(t) = Ve^{j\omega t}$ (here, $j = \sqrt{-1}$)

Solution: $I(t) = Ie^{j\omega t}$

Therefore:

$$-\omega^2 LI + j\omega RI + \frac{I}{C} = j\omega V$$

Giving:

$$I = \frac{j\omega V}{\frac{1}{C} - \omega^2 L + j\omega R} = \frac{j\omega V / L}{\omega_0^2 - \omega^2 + j\omega R / L}$$

Beats & coupled SHM:

$$\cos(\mathbf{w}_1 t) + \cos(\mathbf{w}_2 t) = 2 \cos\left(\frac{\mathbf{w}_1 + \mathbf{w}_2}{2} t\right) \cos\left(\frac{\mathbf{w}_1 - \mathbf{w}_2}{2} t\right)$$

From a trig identity

The second term modulates the beat, and the first provides the 'frequency' of the beat.

Coupled, same mass m :

Coupled pendulums, by a spring

$$m\ddot{x}_1 = -\frac{mg}{l}x_1 - k(x_1 - x_2)$$

$$m\ddot{x}_2 = -\frac{mg}{l}x_2 - k(x_2 - x_1)$$

Adding gives:

$$m(\ddot{x}_1 + \ddot{x}_2) = -\frac{mg}{l}(x_1 + x_2)$$

Say $X = x_1 + x_2$, then:

$$\ddot{X} = -\mathbf{w}_1^2 X \quad \text{which is of the usual form for SHM}$$

With $\mathbf{w}_1 = \sqrt{\frac{g}{l}}$, giving a solution of:

$$X = A_1 \cos(\mathbf{w}_1 t + \mathbf{f}_1) \quad \text{[or in whatever form]}$$

Subtracting initial equations gives:

$$m(\ddot{x}_1 - \ddot{x}_2) = -\frac{mg}{l}(x_1 - x_2) - 2k(x_1 - x_2)$$

And, denoting $x = x_1 - x_2$, gives a solution of:

$$x = A_2 \cos(\mathbf{w}_2 t + \mathbf{f}_2) \quad \text{[or in whatever form]}$$

With $\mathbf{w}_2^2 = \frac{g}{l} + 2\frac{k}{m}$.

X and x are the normal modes of the system – n modes in a system of n coupled bodies.

Rewrite the equations as: $x_1 = \frac{1}{2}(X + x)$ and $x_2 = \frac{1}{2}(X - x)$, so that:

$$x_1(t) = \frac{A}{2}(\cos(\mathbf{w}_1 t) + \cos(\mathbf{w}_2 t)) \quad \text{(if the amplitudes and phases are equal)}$$

So, using the function above for beats, the two expressions can be written as:

$$x_1(t) = A \cos\left(\frac{\mathbf{w}_1 + \mathbf{w}_2}{2} t\right) \cos\left(\frac{\mathbf{w}_1 - \mathbf{w}_2}{2} t\right)$$

$$x_2(t) = A \sin\left(\frac{\mathbf{w}_1 + \mathbf{w}_2}{2} t\right) \sin\left(\frac{\mathbf{w}_1 - \mathbf{w}_2}{2} t\right)$$

Spring with two masses on either end:

$$m\ddot{x}_1 = -k(x_1 - x_2)$$

$$m\ddot{x}_2 = -k(x_2 - x_1)$$

Vibrations & Waves

Then the normal modes are:

$x = x_1 - x_2$ gives $m\ddot{x} = -2kx$, giving an angular frequency $\omega = \sqrt{\frac{2k}{m}}$, with solution of the form $x = A \cos(\omega t + \phi)$

The other mode, $\dot{X} = 0$, gives an un-interesting oscillator system: the centre of gravity proceeds with constant velocity.

Fourier transforms:

For odd, periodic functions:

$$f(\mathbf{q}) = \sum_n a_n \sin(n\mathbf{q}) \quad \text{with:}$$

$$a_n = \frac{1}{P} \int_{-p}^p f(\mathbf{q}) \sin(n\mathbf{q}) d\mathbf{q}$$

For even, periodic functions:

$$f(\mathbf{q}) = \sum_n b_n \cos(n\mathbf{q}) \quad \text{with:}$$

$$b_n = \frac{1}{(1 + \delta_{n0})P} \int_{-p}^p f(\mathbf{q}) \cos(n\mathbf{q}) d\mathbf{q}$$

Generally however, one can express a function as the sum of an even & odd functions:

$$f(\mathbf{q}) = \sum_n a_n \sin(n\mathbf{q}) + b_n \cos(n\mathbf{q})$$

Waves:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Where $v = \sqrt{\frac{T}{m}}$, with m (kg/m) the mass density of a string with some tension T .

- This is for transverse 1-D waves
- Derived using $F = ma$ and tensions on either side of an element.

For longitudinal waves:

e.g. spring, or a metal bar compressed at one end.

Gives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}, \text{ but with } v = \sqrt{\frac{k}{m}}, \text{ for a spring constant } k.$$

- derived using Young's Modulus, and strain/stress

Solutions to wave-equations:

Any $f(x - vt)$ or $f(x + vt)$ is a solution.

The solution $f(x - vt)$ represents a wave travelling in the $+x$ direction, and the other solution is $-x$ travel.

So an example solution would be: $f(x, t) = A \sin(k(x \pm vt)) = A \sin(kx \pm \omega t)$, remembering that $\omega = kv$.

Here, k is the wave number = $\frac{2\pi}{\lambda}$, but as usual, $\omega = 2\pi f$.

Hence, wave motion can be written in any of the following forms:

$$\sin(kx - \omega t) \quad \cos(kx - \omega t) \quad e^{i(kx - \omega t)} \quad \dots$$

Speed of wave given by whatever multiplies x over whatever multiplies t .

$$v = \frac{\omega}{k} = \frac{f\lambda}{f} = \lambda f, \text{ and the speed of a wave on a string is } v = \sqrt{\frac{T}{m}}.$$

Waves at boundaries:

A wave is incident on a boundary – with different mediums:

$$Ie^{i(\omega t - kx)} \quad \& \quad v = \sqrt{\frac{T}{m}} \text{ Before}$$

$$Te^{i(\omega t - k'x)} \quad \& \quad v' = \sqrt{\frac{T}{m'}} \text{ After}$$

Note the same ω and T . These are part of the 'boundary conditions'.

The refractive index is given by:

$$n = \frac{v}{v'} = \frac{\lambda}{\lambda'} = \frac{k'}{k}$$

Another boundary condition is that the join must be smooth.

With just the two waves, Incident & Transmitted, this is not satisfied, so introduce

Reflected wave, hence:

$$Ie^{i(\omega t - kx)} + Re^{i(\omega t + kx)} = Te^{i(\omega t - k'x)},$$

Vibrations & Waves

Evaluating $\frac{\partial}{\partial x}$ on both sides give:

$$-ikIe^{i(\omega t - kx)} + ikRe^{i(\omega t + kx)} = -ik'Te^{i(\omega t - k'x)}$$

Cancelling, and using $x = 0$ at the boundary;

$$I + R = T \quad -ikI + ikR = ik'T$$

Gives some relationships:

$$\frac{T}{I} = \frac{2k}{k+k'} = \frac{2}{1+n} \quad \frac{R}{I} = \frac{k-k'}{k+k'} = \frac{1-n}{1+n}$$

If k and k' are similar, then R is small – as expected.

Standing waves:

Consider a very heavy second rope: v' small k' large, so $T \rightarrow 0$, and $R \rightarrow -I$.

So the wave on the left:

$$Ie^{i(\omega t - kx)} - Ie^{i(\omega t + kx)} = Ie^{i\omega t}(e^{-ikx} - e^{ikx}) = f_t(t)f_x(x), \text{ which, from the function:}$$

$$\sin \mathbf{q} = \frac{e^{iq} - e^{-iq}}{2i}, \text{ one can see that you get } 2 \sin(\omega t) \sin(kx) \text{ out.}$$

Nodes (no disturbance) and anti-nodes (max disturbance) form every half wavelength

$$\sin(kLx) = 0, \text{ with } kL = \mathbf{p}, \mathbf{2p}, \mathbf{3p} \dots \text{ (divided by wavelength).}$$

Only certain wavelengths are allowed, and:

$$\mathbf{w} = \frac{Nv\mathbf{p}}{L} \quad \mathbf{f} = \frac{Nv}{2L}$$

Speed of sound:

Use fact that $pV^g = \text{const}$ hence, $p_0V_0^g = pV^g$

$$\text{So, } p = \frac{p_0V_0^g}{V^g}, \text{ therefore: } \frac{dp}{dV} = -\frac{gp_0V_0^g}{V^{g+1}} = -\frac{gp_0V_0^g}{V \cdot V^g}$$

$$\text{As } p = \frac{p_0V_0^g}{V^g}, \text{ then } \frac{dp}{dV} = -\frac{gp}{V} \Rightarrow dp = -\frac{gp}{V}dV$$

Now, $p = \frac{F}{A}$ and $V = lA$ (volume = length x area), so $dV = A \cdot dl$

Hence, from all the above:

$$\frac{F}{A} = -\frac{gp dV}{V} = -\frac{gp dl A}{lA}$$

$$\therefore F = -\frac{gp dl A}{l} \quad \text{Calling } K = gpA$$

And, as $v = \sqrt{\frac{K}{m}} = \sqrt{\frac{gpA}{m}}$ (remember that $\mathbf{r} = m/V$ and $\mathbf{m} = m/A$)

Therefore: $v = \sqrt{\frac{g}{r}}$

Vibrations & Waves

The $g = \frac{n+2}{n}$, with n the number of degrees of freedom of molecule, so for diatomic

air, $g = \frac{5+2}{5} = 1.4$

The assumptions used in this derivation are:

$$V \propto \frac{1}{p} \quad \text{Constant temperature}$$

Not isothermal changes;

Is adiabatic (no energy flows in or out...);

Air is diatomic.

Standing waves:

Strings, with each end fixed...

$$L = N\lambda/2$$

Pressure:

Displacement:

Pressure node

Displacement node (D-node)

Pressure anti-node

Displacement anti-node

Pipes:

Closed-closed

D-nodes

$$L = N\lambda/2$$

Open-open

D-anti-nodes

$$L = N\lambda/2$$

Closed-open

D-node @ closed end, anti-node @ open

$$L = \lambda/4$$

Energy & power:

For a mass on a spring; $x(t) = a \sin \omega t$

It's KE = $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}ma^2\omega^2 \cos^2 \omega t$

And to find its PE, remember that $F = -kx$:

$PE = -\int Fdx \Rightarrow PE = \frac{1}{2}kx^2 = \frac{1}{2}ka^2 \sin^2 \omega t$

To find total energy = PE + KE, first remember that $\omega^2 = \frac{k}{m} \Rightarrow m\omega^2 = k$

So:

$PE + KE = \frac{1}{2}ka^2(\sin^2 \omega t + \cos^2 \omega t)$
 $= \frac{1}{2}ka^2$

Total energy per unit length, of a *transverse wave*, is given by $E = \frac{1}{2}T\omega^2 a^2$ (its an average).

Actually given by: $T\left(\frac{\partial f}{\partial x}\right)^2$; per unit length.

Remember that $\omega^2 = k^2v^2$ and $v^2 = T/m$.

Total energy per unit length of a *longitudinal wave*, is given by $\frac{1}{2}K\left(\frac{\partial f}{\partial x}\right)^2$.

Energy/pressure in sound waves...

If a displacement = $a \sin(kx - \omega t)$, then the strain = $\frac{\partial f}{\partial x} = ka \cos(kx - \omega t)$.

And, force = $K \times \text{strain} = Kka \cos(kx - \omega t)$.

Also, $K = \rho p_0 A$, so pressure = force/area = $\rho p_0 ka \cos(kx - \omega t)$.

Inserting expressions for k , and things...

Pressure amplitude of oscillation = $p = \frac{2\rho p_0 f a}{v}$

With: a = amplitude, f = frequency, p_0 = atmospheric pressure...

Kinetic energy density is $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho \omega^2 a^2 \cos^2(kx - \omega t)$, as the cosine term gives $\frac{1}{2}$ as an average, and the PE density is equal:

Total average energy density = $\frac{1}{2}\rho \omega^2 a^2$

For a wave traversing (velocity v , in time t) a volume Avt , its energy is = $\frac{1}{2}\rho \omega^2 a^2 Avt$, hence its power density = $\frac{1}{2}\rho \omega^2 a^2 v$

To find decibels... $10 \times \log_{10}\left(\frac{\text{power}}{10^{-12}}\right)$

EM waves:

All the maths leads to the equation: $\frac{\partial^2 E}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$

From Maxwell's equations.

Note that their velocity = $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (in vacuum!)

Spectrum: radio – IR – UV – X-rays – gamma-rays (long – short wavelength)

Polarisation: apply field E_0 , you get a field $E = \frac{E_0}{1 + N\alpha}$

Which all implies a change in the speed of light in a medium: $c = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_0 \mu_r}}$

ϵ_r is the dielectric constant.

Refractive index: $n = c/c' = \sqrt{\epsilon_r \mu_r}$.

EMR travels slower in mediums due to effective weakening of E and B fields.

Group & phase velocity:

$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$
--

A wave described by $\cos(\omega t - kx)$ or $\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$, such that

$v_p = \lambda/T = f\lambda = \omega/k$, and refractive index $n = c/v = c/v_p$

The group velocity is the velocity of the envelope – the information speed.

If two waves: $\omega_1, k_1, f_1, \lambda_1, \omega_2, k_2, f_2, \lambda_2$, then, writing $\omega = \frac{\omega_1 + \omega_2}{2}$ and $k = \frac{k_1 + k_2}{2}$,

and $\Delta\omega = \frac{\omega_1 - \omega_2}{2}$ and similarly for Δk ; and if the waves are superimposed as:

$$e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}, \text{ then:}$$

$$e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}$$

$$= e^{i(kx + \Delta kx - \omega t - \Delta\omega t)} + e^{i(kx - \Delta kx - \omega t + \Delta\omega t)}$$

$$= e^{i(kx - \omega t)} \left[e^{i(\Delta kx - \Delta\omega t)} + e^{i(-\Delta kx + \Delta\omega t)} \right] \qquad \left(\text{Remember that } \cos q = \frac{e^{iq} + e^{-iq}}{2} \right)$$

$$= 2e^{i(kx - \omega t)} \cdot \cos(\Delta kx - \Delta\omega t)$$

The exponential is an infinite wave, velocity $\frac{\omega}{k}$, and the cosine term has velocity $\frac{\Delta\omega}{\Delta k}$.

This implies that the cosine term is the envelope – small wavenumber (k) and frequency, so long wavelength.

If v_p is a constant, then $v_p = v_g$.

Otherwise, $\frac{d\omega}{dk} \neq \frac{\omega}{k}$, giving a refractive index n , which depends upon λ .

So: $\frac{dn}{d\lambda} = \frac{dn}{dk} \cdot \frac{dk}{d\lambda} = \frac{d}{dk} \left(\frac{c}{v_p} \right) \cdot \frac{d}{d\lambda} \left(\frac{2\pi}{\lambda} \right)$; remembering that $v_p = \omega/k$ and $d\omega/dk$ is not a constant.

After algebra, get:

$$v_g = v_p \left(1 + \frac{1}{n} \frac{dn}{d\lambda} \right)$$

Waves in 2/3D:

For 1D, string tension T , density m Kg/m, then:

$$\frac{\partial^2 f}{\partial x^2} = \frac{m}{T} \frac{\partial^2 f}{\partial t^2} \quad \left[v = \sqrt{\frac{T}{m}} \right]$$

In 2D, m Kg/m², then the wave equation becomes:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

To solve, use separation of variables; use $f(x, y, t) = f_x(x)f_y(y)f_t(t)$, and note that $\frac{\partial^2}{\partial x^2}$ only effects f_x . so equation becomes:

$$f_y f_t \frac{\partial^2 f_x}{\partial x^2} + f_x f_t \frac{\partial^2 f_y}{\partial y^2} = \frac{m}{T} f_x f_y \frac{\partial^2 f_t}{\partial t^2}$$

Then, divide by initial function: $f_x f_y f_t$:

$$\frac{1}{f_x} \frac{\partial^2 f_x}{\partial x^2} + \frac{1}{f_y} \frac{\partial^2 f_y}{\partial y^2} = \frac{m}{T} \frac{1}{f_t} \frac{\partial^2 f_t}{\partial t^2}$$

Note now, that the RHS is independent of x, y , and is then a constant: $-k$.

This is similar for the x and y parts of the LHS, call the constants $-k_x^2, -k_y^2$.

Then have 3 equations:

$$\frac{\partial^2 f_a}{\partial a^2} = -k_a^2 f_a \text{ for } a = x, y \text{ and } \frac{1}{v^2} \frac{\partial^2 f_t}{\partial t^2} = -k^2 f_t$$

With solutions $f_a = C_a e^{\pm i k_a a}$ and $f_t = C_t e^{\pm i \omega t}$ remembering that $k^2 = k_x^2 + k_y^2$.

So, $f = f_x f_y f_t = C e^{i(k_x x + k_y y - \omega t)} = C e^{i(k \cdot r - \omega t)}$

With $\mathbf{l}_i = \frac{2\mathbf{p}}{k_i}$

Generalise into 3D:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \nabla^2 f$$

The Doppler effect:

Moving source:

Source moves with speed v . Wave travels at speed c , and it emitted at frequency f .

And wavelength λ .

Then, observed, with primes...

$\lambda' = (1 - \frac{v}{c})\lambda$, and as $c = f\lambda$ then

$$f' = \frac{1}{1 - \frac{v}{c}} \cdot f = \frac{c}{c - v} f$$

Moving observer:

$$f' = (1 + \frac{v}{c})f$$

Relativistic version: $f' = \frac{1}{1 - \frac{v}{c}} f \cdot \frac{1}{\gamma} = \sqrt{\frac{c + v}{c - v}} f$

Huygens construction:

Consider each point on a wavefront as a source of waves, which give another wavefront – but only forward wave needs to be considered – new wavefront is the common tangent.

Cases to be considered:

Plane Wave;

Circular Wave;

Reflection: $\mathbf{q}_i = \mathbf{q}_r$

Refraction: $\frac{\sin \mathbf{q}_i}{\sin \mathbf{q}_r} = \frac{v}{v'} = n$

Single slit: Plane wave strikes a screen, which absorbs everything except one point – waves become circular waves.

Double slit:

Circular wave patterns interfere.

Path difference $d \sin \mathbf{q}$ Constructive interference at $d \sin \mathbf{q} = n\lambda$

Phase difference $\mathbf{d} = \frac{2\mathbf{p}d \sin \mathbf{q}}{\lambda}$

Amplitude received is proportional to $1 + e^{i\mathbf{d}} = e^{\frac{i\mathbf{d}}{2}} \underbrace{\left(e^{-\frac{i\mathbf{d}}{2}} + e^{\frac{i\mathbf{d}}{2}} \right)}_{2\cos \frac{\mathbf{d}}{2}}$

So, intensity = |Amplitude|² $\propto 4\cos^2\left(\frac{\mathbf{d}}{2}\right)$

Note, that $4 = 2^2$, with two slits!

Has peaks at $\frac{\mathbf{d}}{2} = n\mathbf{p}$.

Vibrations & Waves

Many slits: $1 + e^{id} + e^{2id} + \dots + e^{i(N-1)d}$, which is a geometric series, with

$$\text{sum } \frac{e^{iNd} - 1}{e^{id} - 1} = \frac{\sin \frac{Nd}{2}}{\sin \frac{d}{2}}$$

Giving peaks at $d = 2pn$

One wide slit: slit, width a , going from $-a/2$ to $+a/2$. for any q and x , the path

difference is always $\frac{2px \sin q}{\lambda}$

So, total amplitude is proportional to $\int_{-a/2}^{+a/2} e^{i \frac{2px \sin q}{\lambda}} dx$, which eventually

gives a sinc function: $\text{sinc}(u) = \sin(u)/u$.

Polarisation:

Vibrations in horizontal and vertical components (H and V).

Polaroid blocks out one component – so two, at 90deg will block out all.

In reflection of light, more H is reflected than V . Sunglasses work by blocking H , but allowing V to pass.

The colour of the atmosphere is due to scattering – and the light given off is polarised.

Birefringence:

Molecules are polarised (here, it means aligned...) by an E field, giving dielectric constant ϵ_r , and a different speed of light. This effect also separates the V from H components, giving different $\epsilon_r^H, \epsilon_r^V, n_V, n_H$ - so two images are seen.

Circular polarisation arises when the V and H components are out of step, say:

$$E_H = E_0 \cos \omega t \text{ and } E_V = E_0 \sin \omega t, \text{ then } E = \sqrt{E_H^2 + E_V^2} = E_0.$$

Water Waves:

Wave a height h from bottom, to equilibrium position of wave, which is at $y = 0$. Wave velocity c . wave has components v and u in vertical and horizontal directions.

Assumptions:

- 1) nothing happens in z -direction;
- 2) water is incompressible;
- 3) ignore viscosity – no friction/damping;
- 4) amplitudes small: $A \ll \lambda$ and $A \ll h$;
- 5) neglect surface tension;
- 6) vertical acc'n is small – “long waves in shallow water” – tidal waves.

Use pressure differences, areas and differences in height, to derive:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{gh} \frac{\partial^2 f}{\partial t^2} \quad \text{so, } c^2 = gh$$

Solving differential equations:

How to find solutions of the equation:

$$\frac{d^2 f}{dt^2} + b \frac{df}{dt} + cf = 0 \quad \text{Remembering that } f(t), \text{ and } b, c \text{ are constants}$$

So, think of above equation as:

$$\left(\frac{d^2}{dt^2} + b \frac{d}{dt} + c \right) f = 0$$

with: $r_1 r_2 = c$ and $r_1 + r_2 = -b$

$$\Rightarrow \left(\frac{d}{dt} - r_1 \right) \left(\frac{d}{dt} - r_2 \right) f = 0 \quad - (1)$$

So, by the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Now, if you write: $g(t) = \left(\frac{d}{dt} - r_2 \right) f(t)$

$$\text{So, (1) becomes: } \left(\frac{d}{dt} - r_1 \right) g(t) = 0 \Rightarrow \frac{d \cdot g(t)}{dt} = r_1 g(t)$$

Which you can solve by direct integration:

$$\int \frac{dg}{g} = \int r_1 dt \quad \text{giving:} \quad \ln g = r_1 t + \text{const}$$

Therefore:

$$g = e^{r_1 t + \text{const}} = e^{r_1 t} e^{\text{const}} = A e^{r_1 t}$$

So, now need to find f .

$$\text{Now, } \left(\frac{d}{dt} - r_2 \right) f = g = A e^{r_1 t}$$

$$\text{So, } \frac{df}{dt} - r_2 f = A e^{r_1 t}$$

Which can be solved by the *integrating factor* method:

$$I = e^{-\int r_2 dt} = e^{-r_2 t}$$

Multiply by I :

$$e^{-r_2 t} \frac{df}{dt} - r_2 e^{-r_2 t} f = A e^{(r_1 - r_2)t} \quad - (2)$$

Remember that the product rule states:

$$\frac{d}{dt}(uv) = u \frac{dv}{dt} + v \frac{du}{dt}$$

So, (2) becomes:

$$\frac{d}{dt}(e^{-r_2 t} f) = A e^{(r_1 - r_2)t}$$

Which can be integrated, giving:

$$e^{-r_2 t} f = \frac{A}{r_1 - r_2} e^{(r_1 - r_2)t} + B$$

Which you can express as:

$$e^{-r_2 t} f = C e^{(r_1 - r_2)t} + B \quad \text{if } r_1 \neq r_2$$

So, therefore:

$$f(t) = C e^{r_1 t} + D e^{r_2 t}$$

Which is the solution!

Note, that for r_1, r_2 it doesn't matter which we choose for the positive or negative sign!

How to find solutions of the equation:

$$\frac{d^2 f}{dt^2} + b \frac{df}{dt} + cf = F(t) \quad - (3)$$

Now, suppose f_{PI} solves (3) (particular integral)

And f_{CF} solves $\frac{d^2 f}{dt^2} + b \frac{df}{dt} + cf = 0$ (the complementary function)

So, $(f_{PI} + f_{CF})$ also solves (3); because:

$$\left(\frac{d^2}{dt^2} + b \frac{d}{dt} + c \right) (f_{PI} + f_{CF}) = F(t) + 0 = F(t)$$

So, any particular integral, plus the complementary function will give a general solution.

Suppose $F(t) = F_0 \cos(\Omega t)$ and $f \sim e^{i\Omega t}$

So, $f_{PI}(t) = A e^{i\Omega t}$, with $A = \frac{F_0/m}{\Omega^2 - \omega^2 + i g \omega}$ which contains no arbitrary constants!

So, the complete function: $f = f_{complete} = f_{PI} + f_{CF}$

$$f_{complete} = A e^{i\Omega t} + \underbrace{B e^{-\frac{\gamma}{2} + i\omega_0 t} + C e^{-\frac{\gamma}{2} t - i\omega_0 t}}_{CF}$$

Note, the $A e^{i\Omega t}$ has no damping term – is not damped.

The B and C terms have gamma's – they are damped.

After time $t \gg \frac{1}{g}$ they vanish!

These are the *transients*.

At large t , can forget about the transients.

But cannot at small times.

$$F(t) = F_0 \cos \Omega t \quad t \geq 0$$

$$F(t) = 0 \quad t < 0$$

$f(t) = 0$ at $t = 0$ and before.

$f'(t) = 0$ at $t = 0$

so, can rewrite $f_{complete}$ as:

$$f_{complete} = A e^{i\Omega t} + e^{-\frac{i\gamma}{2}} (B e^{i\omega_0 t} + C e^{-i\omega_0 t})$$

Can find B, C by the initial conditions.

Matter waves:

Photoelectric effect: waves come in quanta of specific energies... $E = hf$ or $E = \hbar\omega$

From relativity: $E^2 = p^2c^2 + m_0^2c^4$ for photons $m_0 = 0$.

So $E = pc \Rightarrow p = E/c = hf/c = h/\lambda = \hbar k$.

And, for Compton Scattering... photons energy E scatter off electrons.

E', p' are for the scattered photon, and E_e, p_e for the scattered electron.

Then, conservation of momentum says: $\underline{p} = \underline{p}' + \underline{p}'_e$

And conservation of energy: $E + m_e c^2 = E' + E_e$

After loads of algebra, get:

$$E' = \frac{m_e c^2 E}{m_e c^2 + E(1 - \cos\theta)}$$

This lot implies that we should treat photons as particle-like: the usage of $E = hf$.

So, how about particles behaving like waves...?

If particles are waves, they need a wavelength and frequency: $f = E/h, \lambda = h/p$

Or, $E = \hbar\omega$ & $p = \hbar k$

So, we should be able to see diffraction and interference effects, if dimensions of 'slit' are comparable to the wavelength.

Which is of the order of $\lambda \sim 10^{-34}m$ for a 1kg ball.

But, for an electron: $\lambda \sim 10^{-3}m$.

There is a little confusion when talking about group/phase velocities, but if you stick to the particle velocity as being the group velocity, everything is fine.

Now, waves can be described by a wave equation: $\mathbf{y}(x,t) = e^{i(kx - \omega t)}$.

Which describes a free particle wave – no boundaries/refraction...

So what about interacting particles...?

Suppose they are described by some $\mathbf{y}(x,t)$... what do we know about p, E ?

From the original wave equation: $\mathbf{y}(x,t) = e^{i(kx - \omega t)}$, to get $\hbar\omega, \hbar k$:

$$\left(i\hbar \frac{\partial}{\partial t} \right) \mathbf{y} = \hbar\omega \mathbf{y} = E \mathbf{y}$$

$$\left(-i\hbar \frac{\partial}{\partial x} \right) \mathbf{y} = \hbar k \mathbf{y} = p \mathbf{y}$$

So, if we write: $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ and $\hat{E} = i\hbar \frac{\partial}{\partial t}$ which are just operators.

So, we have:

$$\hat{p} \mathbf{y} = p \mathbf{y} \quad \text{and} \quad \hat{E} \mathbf{y} = E \mathbf{y}$$

$$\text{Usually: } E = PE + KE = V(x) + \frac{p^2}{2m} \quad \left\{ \frac{1}{2}mv^2 = \frac{p^2}{2m} \right\}$$

$$\text{So: } E \mathbf{y} = V(x) \mathbf{y} + \frac{p^2}{2m} \mathbf{y}$$

$$\text{Therefore: } -\frac{\hbar^2}{2m} \frac{\partial^2 \mathbf{y}(x,t)}{\partial x^2} + V(x) \mathbf{y} = i\hbar \frac{\partial \mathbf{y}(x,t)}{\partial t}$$

Which is the Schrödinger wave-equation for matter.