Electricity:

Basics:

Force on point charge Q due to point charge q:

$$\underline{F}_Q = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathfrak{R}^2}\hat{\underline{\mathfrak{R}}}$$

Where $\hat{\underline{\mathfrak{R}}}$ is a unit vector from source to test: q to Q.

Principle of superposition:

$$\underline{F}_Q = \underline{F}_1 + \underline{F}_2 + \ldots$$

So, $\underline{F}_Q = \frac{Q}{4\pi\varepsilon_0}\left[\frac{q_1}{\mathfrak{R}_1^2}\hat{\underline{\mathfrak{R}}}_1 + \frac{q_1}{\mathfrak{R}_2^2}\hat{\underline{\mathfrak{R}}}_2 + \ldots\right] = Q\underline{E}$

So, $\underline{F} = Q\underline{E}$

With:

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\mathfrak{R}_i^2}\hat{\underline{\mathfrak{R}}}_i$$

Or, for a continuous charge distribution, use charge densities:

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathfrak{R}_i^2}\hat{\underline{\mathfrak{R}}}_i dq$$

Such that: $dq = \lambda dl$, $dq = \sigma ds$, $dq = \rho dV$ depending on the geometry.

Flux is the amount of stuff though an area, so for electric fields:

$$\phi = \int \underline{E}\cdot d\underline{S}$$

Gauss' law:

$$\int_S \underline{E}\cdot d\underline{S} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho dV$$

V is some volume, enclose by an arbitrary surface S.

Depending on the symmetry of the problem, different 'Gaussian Surfaces' can be used:

Spherical symmetry:	choose S to be a sphere
Cylindrical symmetry	choose S to be a cylinder, around the wire
Planar-rectangular symmetry	choose a cylinder, going through the plane, remembering that there are two ends of the cylinder!

Circulation & Potential:

Circulation of an $\underline{E}$ field is zero:

$$\oint_l \underline{E} \cdot d\underline{l} = 0$$

But, the electric potential is defined:

$$-\int_A^B \underline{E} \cdot d\underline{l} = \phi(\underline{r}_B) - \phi(\underline{r}_A) = \Delta\phi_{AB}$$

Which is the energy needed to move unit charge from *A* to *B*.

More generally, choose infinity as the reference point, so the electric potential at *r*:

$$\phi(\underline{r}) = -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

Call *V* the potential… superposition works: $V = \sum_i V_i$

Can denote potential by other symbols, such as phi, as is done above & below.

Now, $dV = \dfrac{dq}{4\pi\epsilon_0\Re}$ and integrate.

Potential is a scalar quantity.

$$\phi = -\int_\infty^r \underline{E} \cdot d\underline{l}$$

$$\underline{E} = -\underline{\nabla}\phi$$

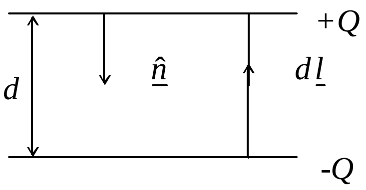
Conductors & Capacitance:

Electric field = 0 inside a conductor.
Charge resides on the surface of a conductor.
$E_{outside}$ is perpendicular to the surface of the conductor.

Capacitance:

$$C = Q/V$$

For a parallel plate capacitor:



σ = charge/m^2

As previously derived, $\underline{E} = \dfrac{\sigma}{\epsilon_0}\hat{\underline{n}}$

Also, $V = -\int_{-}^{+} \underline{E} \cdot d\underline{l}$

So: $V = \int_{-}^{+} E dl = E \int dl = Ed = \dfrac{s d}{e_0} = \dfrac{Qd}{Ae_0}$

And, $C = \dfrac{Q}{V} = \dfrac{e_0 A}{d}$

Work:

When charge moved from one plate to another, work is done:

$$dW = Vdq = \frac{q}{C} dq \Rightarrow W = \int dW = \frac{1}{C} \int_0^Q qdQ = \frac{1}{2} \frac{Q^2}{C}$$

$$\boxed{W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C}}$$

So, for a parallel plate capacitor:

$$W = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{(sA)^2}{e_0 A / d} = \frac{1}{2} \frac{s^2}{e_0} Ad = \frac{1}{2} e_0 E^2 \times volume$$

So,

$$\boxed{\text{Energy density in electric field} = \frac{1}{2} e_0 E^2} \qquad [\text{J/m}^3]$$

Dipoles:

Dipole moment: $\underline{p} = q\underline{d}$

Which is from 2 opposite charges, a distance *d* apart.

After loads of algebra, get:

$$f(r) = \frac{1}{4pe_0} \frac{\underline{p} \cdot \underline{r}}{r^3} = \frac{1}{4pe_0} \frac{\underline{p} \cdot \hat{\underline{r}}}{r^2}$$

The dipole field for a perfect dipole looks like this:

Magnetism:

Basics:

Lorentz force law: $\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$

$\underline{B}$-fields do circulate around a wire:

$\oint \underline{B} \cdot d\underline{l} \neq 0$

And:

$\int \underline{B} \cdot d\underline{S} = 0$ No magnetic monopoles, and magnetic fields NEVER do any work!

Current = charge flowing past a point per second.

λ per m of charge, travelling at $\underline{v}$:

$$\Rightarrow \underline{I} = \frac{\lambda \underline{v} \Delta t}{\Delta t} = \lambda \underline{v}$$

For a small segment: $d\underline{F} = \left(\underline{v} \times \underline{B}\right) dq \Rightarrow \underline{F} = \int \left(\underline{v} \times \underline{B}\right) dq$

For a charge density $\lambda \Rightarrow dq = \lambda \, dl$

So: $\underline{F} = \int \lambda \left(\underline{v} \times \underline{B}\right) dl = \int \left(\underline{I} \times \underline{B}\right) dl = \int I \left(d\underline{l} \times \underline{B}\right) = \int IB \sin\theta \, dl \, \underline{\hat{n}}$

Then, if $\underline{B}$ is at right angles to $\underline{I}$:

$$\underline{F} = BIl \, \underline{\hat{n}}$$

If $\underline{j}$ = current flowing through a unit area, then:

$$\underline{I} = \int_S \underline{j} \cdot d\underline{S}$$

Biot-Savart Law:

$$B\left(\underline{r}\right) = \frac{\mu_0}{4\pi} \int_l \frac{\underline{I} \times \underline{\hat{\Re}}}{\Re^2}$$

Or, for steady current ONLY:

$$B\left(\underline{r}\right) = \frac{\mu_0 I}{4\pi} \int_l \frac{d\underline{l} \times \underline{\hat{\Re}}}{\Re^2}$$

To find the magnetic field around a wire:

- cylindrical symmetry.

$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{d\underline{l} \times \underline{\hat{\Re}}}{\Re^2}$$

Each pair of elements give an overall field along z only.
So:

$$\underline{B} = \int dB_z \hat{z}$$

$d\underline{l}$ and $\hat{\Re}$ are at right-angles, so: $\quad d\underline{l} \times \underline{\Re} = dl \sin\theta$

$$\Rightarrow dB_z = \frac{\mu_0 I}{4\pi} \frac{dl}{\Re^2} \sin\theta$$

Looking at the loop head-on:

$$d\underline{l} = s\,d\phi$$

$$\therefore dB_z = \frac{\mu_0 I}{4\pi \Re^2} \sin\theta s\,d\phi$$

$$\Rightarrow B_z = \frac{s\mu_0 I \sin\theta}{4\pi \Re^2} \int_0^{2\pi} d\phi = \frac{\mu_0 I s \sin\theta}{2\Re^2}$$

Notice that $\Re^2 = s^2 + z^2$ and $\sin\theta = \frac{s}{\Re} = \frac{s}{\left(s^2 + z^2\right)^{\frac{1}{2}}}$

Therefore:

$$\underline{B} = \frac{\mu_0 I}{2} \frac{s^2}{\left(s^2 + z^2\right)^{\frac{3}{2}}} \underline{\hat{z}}$$

To find the magnetic field around a straight wire:

Field circulates around wire.

So:

$$\underline{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\underline{l} \times \underline{\hat{\Re}}}{\Re^2}$$

Evaluating the cross-product gives: $\quad \underline{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{\sin\alpha}{\Re^2} dl$

Now, from the diagram, $\sin\alpha = \cos\theta$

Note, $s = \Re\cos\theta \Rightarrow \frac{1}{\Re} = \frac{\cos\theta}{s}$

$$l = s\tan\theta \Rightarrow dl = \frac{s}{\cos^2\theta} d\theta$$

So: $B = \frac{\mu_0 I}{4\pi} \int \cos\theta \frac{s}{\cos^2\theta} \frac{\cos^2\theta}{s^2} d\theta = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_0 I}{4\pi s} [\sin\theta]_{\theta_1}^{\theta_2}$

For an infinite straight wire: $B = \frac{m_0 I}{4ps}\left[\sin q\right]_{-\frac{p}{2}}^{\frac{p}{2}} = \frac{m_0 I}{2ps}$

So: $\underline{B} = \frac{m_0 I}{2ps}\underline{\hat{q}}$ in cylindrical polars.

Ampere's Law:

$$\oint_l \underline{B}\cdot d\underline{l} = m_0 I_{enc} = \int_S \underline{j}\cdot d\underline{S}$$

Closed loop *l* defines any area *S*.

An example, to find the magnetic field around a long straight wire:

"$\underline{B}$ circulates around the wire"

Choose loop: $\oint \underline{B}\cdot d\underline{l} = \oint Bdl = B\oint dl = 2prB$

And, from Ampere's Law: $B2pr = m_0 I \Rightarrow \underline{B} = \frac{m_0 I}{2pr}\underline{\hat{q}}$

As before… but this method is easier!

In a solenoid, no current inside, or outside – only on the surface.

EMF & Faraday/Lenz's Law:

EMF (e) – work needed per unit charge to maintain a current.

$e = -\int_b^a \underline{E}\cdot d\underline{l}\left(= V_{ab}\right)$ generally: $e = \oint \underline{F}\cdot d\underline{l}$

Motional EMF:

$\underline{F} = q(\underline{v}\times\underline{B})$ so, per unit charge: $\underline{F} = \underline{v}\times\underline{B}$

So: $e = \int_a^b (\underline{v}\times\underline{B})\cdot d\underline{l} = vB\int dl = vBl$

Also, show by rate of change of *L* (above diagram) that:

$$e = -\frac{df}{dt}$$

EMF per coil!

- electromagnetic induction!

$$\oint_l \underline{E}\cdot d\underline{l} = e = -\frac{df}{dt} = -\frac{d}{dt}\int_S \underline{B}\cdot d\underline{S}$$

surface *S*, bound by loop *l*.

Which says:
Induced EMF in a closed loop is equal to the rate of change of magnetic flux through the loop.

The actual, time dependant version of Ampere's law is:

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 \int_S \left(\underline{j} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \cdot d\underline{S}$$

But the quasi-static approximation neglects the $\partial \underline{E} / \partial t$ part.

Inductance:

As $B \propto I$ and $\phi \propto B$ therefore $\phi \propto I$

$$\phi_2 = M_{21} I_1$$

M_{21} is the mutual inductance… depends only on the geometry of the circuit.

So: $$\varepsilon_2 = -\frac{d\phi_2}{dt} = -M_{21} \frac{dI_1}{dt}$$

$M_{21} = M_{12} = M$ generally.

Example of two solenoids:
Field: $\underline{B} = I\mu_0 n \hat{\underline{z}}$ inside, and 0 outside.
n = #turns per unit length.
They have an overlap length of l.
Wound in the same sense.
Radius s.

Flux through one turn of (2) due to (1):

$$\phi_2 = \int_{S_2} \underline{B}_1 \cdot d\underline{S}_2 = B_1 \pi s^2$$

There are $n_2 l$ turns in solenoid (2)

$$\Rightarrow \phi_2 = B_1 \pi s^2 n_2 l = \underbrace{\mu_0 n_1 n_2 \pi s^2 l}_{M} I_1$$

If they were wound in opposite sense: $M = -\mu_0 n_1 n_2 \pi s^2 l$

In transformers:

$\phi_1 \propto N_1$

$\phi_2 \propto N_2$

$$\therefore \frac{\phi_1}{\phi_2} = \frac{N_1}{N_2} \Rightarrow N_2 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_2}{dt}$$

Therefore:

$N_2 \varepsilon_1 = N_1 \varepsilon_2$ so: $$\boxed{\frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2}}$$

Self-Inductance:

If the current changes, the magnetic field changes, so the flux changes, and the EMF opposes the change in the current.
This gives you a back EMF.
Self inductance L

$f = \int_S \underline{B} \cdot d\underline{S} \propto I$ so: $f = LI$

And $\frac{df}{dt} = L\frac{dI}{dt}$

Therefore:

$$e = -L\frac{dI}{dt}$$

Example of a single solenoid:

One turn: $f_1 = \int \underline{B} \cdot d\underline{S} = BpR^2$

N turns: $f = Nf_1 = BpR^2N = BpR^2nl$

Putting in the expression for the magnetic field:

$f = \underbrace{m_0pn^2R^2l}_{L}I$

L is always positive.

Work:

In order to get a current going, you need to supply work to get charges to move against the back EMF:

$dW = -edq = -eIdt = L\frac{dI}{dt}Idt = LIdI$

$\Rightarrow W = L\int_0^I IdI = \frac{1}{2}LI^2$

$$W = \frac{1}{2}LI^2$$

So, for the solenoid: $B = m_0 nI \Rightarrow I = \frac{B}{m_0 n}$

$L = m_0pn^2R^2l = m_0n^2 \times volume$

So, $W = \frac{1}{2}LI^2 = \frac{1}{2}m_0n^2\frac{B^2}{m_0^2} \times volume$

So, the energy density in a solenoid, or any geometry:

$$= \frac{1}{2m_0}B^2$$

Total energy stored: $\int_V \frac{1}{2m_0}B^2 dt$