Moment of Inertia Tensor:

$$I_{ij} = m_a \left(\underline{r}_a^2 d_{ij} - r_{a,i} r_{a,j}\right)$$

Where: $\underline{r}_a = r_{a,1}\hat{\underline{e}}_1 + r_{a,2}\hat{\underline{e}}_2 + r_{a,2}\hat{\underline{e}}_3$

Such that: $\underline{r}_a^2 = r_{a,1}^2 + r_{a,2}^2 + r_{a,3}^2$

Making use of: $r = \frac{m}{V} = \frac{dm}{dV} \Rightarrow dm = r\,dV$

For a body with a -elements.

This gives the elements of the tensor:

$$\underline{\underline{I}} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

The evaluation of the elements can be simplified in a number of ways:

Perpendicular axis theorem: $I_{11} + I_{22} = I_{33}$

To derive this:

Take a 2D object, with surface mass-density $s(\underline{r})$, in the x_1, x_2 plane.

So: $I_{ij} = \int_A s(\underline{r})\left(\underline{r}^2 d_{ij} - r_i r_j\right) dS$

Choose $x_3 = 0$ for all points in the plane – symmetry axis.

This quickly gives: $I_{13} = I_{31} = 0 \;\&\; I_{23} = I_{32} = 0$

Now:

$$I_{11} = \int_A s(\underline{r})\left(r_1^2 + r_2^2 - r_1^2\right) dS = \int_A s(\underline{r}) r_2^2\, dS$$

$$I_{22} = \int_A s(\underline{r}) r_1^2\, dS$$

$$I_{33} = \int_A s(\underline{r})\left(r_1^2 + r_2^2\right) dS \qquad \Rightarrow I_{33} = I_{11} + I_{22}$$

Now, the principle axes of inertia have the property that $\underline{L}$ & $\underline{w}$ are parallel for rotations about these principle axes. Suppose that $\underline{e}_i$ is a principle axis. Then:

$\underline{\underline{I}}\,\underline{e} = I\underline{e}$, showing that $\underline{e}$ is an eigenvector, with I an eigenvalue.

This will give a diagonal tensor, with elements the 'principle moments of inertia'. The principle axes of a body are also its axes of symmetry.

$\underline{L} = \underline{\underline{I}} \cdot \underline{w}$	Angular momentum
$T = \frac{1}{2}\underline{w} \cdot \underline{L} = \frac{1}{2}\underline{w} \cdot \left(\underline{\underline{I}} \cdot \underline{w}\right)$	Kinetic energy of rotation
$\underline{t} = \underline{w} \times \underline{L}$	Torque

Eulers Equations:

We have: $\underline{t} = \left(\frac{d\underline{L}}{dt}\right)_{rot} + \underline{w} \times \underline{L}$ (see non-inertial reference frame stuff)

Eventually giving:

$\underline{t} = \underline{\underline{I}}\dot{\underline{w}} + \underline{w} \times \left(\underline{\underline{I}}\underline{w}\right)$

Which, after evaluating all the cross-products gives Euler's equations:

$$\begin{aligned} t_1 &= I_1\dot{w}_1 + (I_3 - I_2)w_2 w_3 \\ t_2 &= I_2\dot{w}_2 + (I_1 - I_3)w_1 w_3 \\ t_3 &= I_3\dot{w}_3 + (I_2 - I_1)w_1 w_2 \end{aligned}$$

These can be applied:

Suppose $\underline{t} = 0$, and the principle axes of inertia are such that $I_1 = I_2 = I$. It can then be easily shown that $\dot{w}_3 = 0$. Then you are left with:

$$\begin{cases} I\dot{w}_1 + (I_3 - I)w_2 w_3 = 0 \\ I\dot{w}_2 + (I - I_3)w_1 w_3 = 0 \end{cases}$$

If you define $\Gamma = \left(\frac{I_3 - I}{I}\right)w_3$, then the above equations simplify easily to:

$$\begin{cases} \dot{w}_1 + \Gamma w_2 = 0 \\ \dot{w}_2 - \Gamma w_1 = 0 \end{cases} \quad \Rightarrow \quad \begin{aligned} \dot{w}_1 &= -\Gamma w_2 \\ \dot{w}_2 &= \Gamma w_1 \end{aligned}$$

So, differentiating:

$$\begin{cases} \ddot{w}_1 + \Gamma \dot{w}_2 = 0 \\ \ddot{w}_2 - \Gamma \dot{w}_1 = 0 \end{cases}$$

Inserting the above expressions for $\dot{w}_1, \dot{w}_2$:

$$\begin{cases} \ddot{w}_1 + \Gamma^2 w_1 = 0 \\ \ddot{w}_2 + \Gamma^2 w_2 = 0 \end{cases}$$

Which are just expressions for SHM, assuming that $\Gamma^2 > 1$ (otherwise get 'complex' motion involving hyperbolics!).

So: $\begin{aligned} w_1 &= A\cos(\Gamma t + f) \\ w_2 &= A\sin(\Gamma t + f) \end{aligned}$ as $\dot{w}_1 = -\Gamma w_2$

With appropriate constants for amplitude & phase difference.

Gravitation:

$$\underline{F} = -\frac{GMm}{r^2}\hat{\underline{e}}$$ ($\hat{\underline{e}}$ here is a unit vector in the opposite direction to the central body)

Keplers laws of planetary motion:
1) Law of ellipses: Orbit of each planet is an ellipse, with the Sun at one foci.
2) Law of equal areas: A line drawn between the Sun and a planet sweeps out an equal area in a equal time.
3) Harmonic law: $T^2 \propto R^3$

Conservative forces: work done by the force is independent of the path.
For conservative forces *only*, there is a potential energy function *U*($\underline{x}$), such that:
$\underline{F} \cdot d\underline{x} = -dU$, from which you can show:

$$\boxed{\underline{F} = -\nabla U}$$

Reduced mass: $m = \frac{m_1 m_2}{m_1 + m_2}$

And, Newton: $m\ddot{\underline{r}} = f(\underline{r})$

Conservation of angular momentum: $\frac{dA}{dt} = \frac{L}{2m}$

$$r = \frac{a}{1 + e\cos q} \qquad e = \sqrt{1 + \frac{2h\,l^2}{G^2M^2}} \qquad l = \frac{L}{m} \quad h = \frac{E}{m}$$

$$a = \frac{l^2}{GM} \qquad T = \frac{2pa^{\frac{3}{2}}}{\sqrt{GM}}$$

Total energy: $-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r} = E$

If you want to work out the period of a circular orbit, say, remember that $F = mrw^2$ and that $w = 2pf = \frac{2p}{T}$ and that $F = \frac{GMm}{r^2}$, then rearrange.

Non-Inertial Reference Frames:

For any position vector $\underline{W}$ in a rotating reference frame:

$$\frac{d\underline{W}}{dt} = \left.\frac{d\underline{W}}{dt}\right|_{rot} + \underline{w} \times \underline{W}$$

For example, if a position vector $\underline{r}$, then the velocity vector $\underline{v}$ is given by:

$$\underline{v} = \frac{d\underline{r}}{dt} = \left.\frac{d\underline{r}}{dt}\right|_{rot} + \underline{w} \times \underline{r} \Rightarrow \underline{v} = \underline{v}' + \underline{w} \times \underline{r}$$

Do it again for acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = \left.\frac{d}{dt}\left(\underline{v}' + \underline{w} \times \underline{r}\right)\right|_{rot} + \underline{w} \times \left(\underline{v}' + \underline{w} \times \underline{r}\right) \qquad (\text{Assume } \dot{w} = 0)$$

So:

$$\underline{a} = \underline{a}' + 2\underline{w} \times \underline{v}' + \underline{w} \times \left(\underline{w} \times \underline{r}\right)$$

And, from $\underline{F} = m\underline{a}$:

$$\boxed{F - 2m\underline{w} \times \underline{v}' - m\underline{w} \times \left(\underline{w} \times \underline{r}\right) = m\underline{a}'}$$

So:

Coriolis force is the $2m\underline{w} \times \underline{v}'$ part.

Centrifugal force is the $m\underline{w} \times \left(\underline{w} \times \underline{r}\right)$ part.

If you define:

$\underline{e}_1$ as North

$\underline{e}_2$ as Up

$\underline{e}_3$ as East

l as latitude

Then, if: $\underline{x} = 0\underline{e}_1 + 0\underline{e}_2 + R\underline{e}_3$

$$w = \left(\cos l\, \underline{e}_2 + \sin l\, \underline{e}_3\right)w = w\left(0, \cos l, \sin l\right)$$

Then all the above forces can be found by evaluating the cross products.

Relativity:

Lorenz transformations:

$x_1' = g(v)(x - vt)$

$t' = g(v)\left(t - \frac{vx}{c^2}\right)$ $\qquad g(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Velocity transformations:

$V_x = \frac{dx}{dt} = \frac{V_x' + v}{1 + \frac{vV_x'}{c^2}}$ $\qquad V_y = \frac{dy}{dt} = \frac{V_y'}{g\left(1 + \frac{vV_x}{c^2}\right)}$

$E = g(v)mc^2$ $\quad \underline{p} = g(v)m\underline{v}$ both imply that: $\quad E^2 = p^2c^2 + m^2c^4$

Define $\underset{\sim}{x} = (\underline{x}, ict)$ a 4-vector.

Now, this is true: $\quad \underset{\sim}{x} \cdot \underset{\sim}{x} = \underset{\sim}{x}' \cdot \underset{\sim}{x}'$ $\quad$ -invariance

Now, in 3D: $\quad x_i' = R_{ij}x_j$

So, in the 4D version: $\quad x_m' = L_{mn}x_n$ so $\quad \underset{\sim}{x}' = \underline{\underline{L}}\underset{\sim}{x}$

You can think of $\underline{\underline{L}}$ as a 4D 'rotation' matrix.

So: $\quad \underset{\sim}{x} \cdot \underset{\sim}{x} = x_m x_m = \underline{x}^2 - c^2t^2 = -c^2\tau^2$

Where we have defined the proper time:

$\tau^2 = (t_1 - t_2)^2 - \frac{1}{c^2}(\underline{x}_1 - \underline{x}_2)^2$

We can define a 4-velocity: $\quad \underset{\sim}{u} = \frac{d\underset{\sim}{x}}{d\tau} = g(u)(\underline{u}, ic)$

Therefore:

$$\underline{\underline{L}} = \begin{pmatrix} g & 0 & 0 & \frac{igv}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{igv}{c} & 0 & 0 & g \end{pmatrix}$$

Which can be used to derive the velocity transformations.

The wave 4-vector: $\quad \underset{\sim}{k} = \left(\underline{k}, \frac{iw}{c}\right)$

Momentum 4-vector: $\quad \underset{\sim}{p} = g(v)(\underline{p}, imc)$

$\underset{\sim}{p} \cdot \underset{\sim}{p} = -m^2c^2$