Probability Distributions:

P_i is the probability of outcome x_i.

$< x >= \sum_i P_i x_i$ is the expectation value (or "average") $= \bar{x}$

Standard deviation= s .

Variance:

$$
\begin{aligned}
\mathrm{var}(x) = s^2 &=< (x - < x >)^2 > \\
&=< x^2 - 2x < x > + < x >^2 > \\
&= \sum_i P_i x_i^2 - 2 < x > \sum_i P_i x_i + < x >^2 \underbrace{\sum_i P_i}_{=1} \\
&=< x^2 > -2 < x >^2 + < x >^2 \\
&=< x^2 > - < x >^2
\end{aligned}
$$

$$\therefore\ s^2 =< x^2 > - < x >^2$$

Geometric Distributions:

Probability of success on "first go" = p.

So prob. of failure = $1 - p = q$.

$$P_n = pq^{n-1}$$

Need that this distribution adds to unity. i.e. that $\sum_{n=1}^{\infty} P_n = 1$:

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_n &= p + pq + pq^2 + \ldots \\
&= p + q(p + pq + pq^2 + \ldots) \\
&= p + q \sum P_n
\end{aligned}
$$

$\therefore \quad \sum P_n = p + q \sum P_n$

$\Rightarrow \quad \sum P_n - q \sum P_n = p$

$\therefore \quad \sum P_n (1 - q) = p$

$\therefore \quad \sum P_n = \frac{p}{1-q}$

But: $\quad p = 1 - q$

$\Rightarrow \quad \sum_{n=1}^{\infty} P_n = \frac{p}{1-q} = \frac{p}{p} = 1$

See that the average now is given by $< n >= \sum_{n=1}^{\infty} nP_n$. To compute this value:

$$< n >= \sum_{n=1}^{\infty} nP_n = \sum npq^{n-1} = p \sum nq^{n-1}$$

Notice that:

$$nq^{n-1} = \frac{d}{dq}\left(q^n\right)$$

$$\Rightarrow \qquad \sum nq^{n-1} = \frac{d}{dq}\left(\sum q^n\right)$$

But, $\sum q^n$ is the sum of a geometric series.
So:

$$\sum nq^{n-1} = \frac{d}{dq}\sum q^n = \frac{d}{dq}\left(\frac{q}{1-q}\right)$$
$$= \frac{1}{1-q} + \frac{q}{(1-q)^2}$$
$$= \frac{1}{(1-q)^2}$$

$$\Rightarrow \qquad <n> = p\sum nq^{n-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\therefore \qquad <n> = \frac{1}{p}$$

To find the variance, calculate $<n^2>$, and substitute in. To do this:
See that you need to find the expression for:

$$\sum n^2 P_n = \sum n^2 pq^{n-1}$$
$$= p\sum n^2 q^{n-1}$$
$$= p\sum n.\underbrace{nq^{n-1}}_{=\frac{d}{dq}(q^n)}$$
$$= p\sum n\frac{d}{dq}(q^n)$$
$$= p\frac{d}{dq}\sum nq^n$$
$$= p\sum n(n-1)q^{n-1}$$

Now, taking out q:

$$\therefore \qquad \sum n^2 P_n = pq\sum n(n-1)q^{n-2} = \sum (n^2 - n)P_n$$

And, see that:

$$\sum n(n-1)q^{n-2} = (-1)\left(-\frac{2}{(1-q)^3}\right)$$

$$\therefore \qquad pq\sum n(n-1)q^{n-2} = \frac{2pq}{(1-q)^3}$$

Hence:

$$\sum (n^2 - n)P_n = \frac{2pq}{(1-q)^3}$$

Notice that:

$$\sum (n^2 - n)P_n = \sum n^2 P_n - \sum nP_n = <n^2> - <n>$$

Therefore, using the above expression for $<n>$:

$$<n^2> = <n> + \frac{2pq}{(1-q)^3} = \frac{1}{p} + \frac{2(1-p)}{p^2}$$

Hence, the variance var*(n)* can be computed:

$$\mathrm{var}(n) = <n^2> - <n>^2$$

$$= \left(\frac{1}{p} + \frac{2(1-p)}{p^2} \right) - \left(\frac{1}{p} \right)^2$$

$$= \frac{1-p}{p^2}$$

Continuous Distributions:

Suppose a discrete prob.dist has a resolution down to Δx, so that $P_i = P(x_i < x < x_i + \Delta x)$.
Let $\Delta x \to 0$, so that:

$$P(x_i < x < x_i + \Delta x) = f(x_i)\Delta x$$

Hence:

$$P(a < x < b) = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x = \int_a^b f(x)dx$$

With an expectation value $< x >$:

$$< x > = \int_{-\infty}^{+\infty} x f(x)dx$$

Notice that:

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

The variance and standard deviation is defined as it was before, but with the integral sign instead of a summation sign.

Gaussian Distributions:

Displayed as a bell-shaped graph. With the "peak" as the average m. The standard deviation s will show points a little to the right and left of m. They are usually given by functions of the form:

$$f(x) = \frac{e^{-(x-m)^2/2s^2}}{\sqrt{2p s^2}}$$

"Cumulative" functions *P(x)* and *C(x)* are to show probabilities of the function – for example, $P(m+s)$ gives the probability that a result lies *above* the $m+s$ value. And similarly $C(m-s)$ gives the probability below the point $m+s$. The main point of these is also that $P(m+s) = C(m-s)$; from the symmetry of the bell-curve.
Also, $P(m+2s) = C(m-2s)$. And so on.
These cumulative probabilities are defined by:

$$C(X) \equiv P(Y < X) = \int_{-\infty}^{X} f(y)dy$$

The P is not the above cumulative function *P(x)*; it is a probability function.
P(x) can also be thought of as a survival probability:

$$P(X) \equiv 1 - C(X)$$

$$= \int_{X}^{+\infty} f(y)dy$$

An important integral result for Gaussian's is below:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \left(\frac{p}{a}\right)^{\frac{1}{2}}$$

An example of a Geometric Distribution:

Suppose you have an a particle, making F collisions per second, with a barrier it is trying to get out of. In a time t it has Ft attempts. Each attempt has probability p of success, so probability of failure = *(1 - p)*. So, the probability that the nucleus survives to time t is:

$$\begin{aligned} P(t) &= (1-t)^{Ft} \\ &= \left(e^{\ln(1-p)}\right)^{Ft} \\ &= \left(e^{F\ln(1-p)}\right)^{t} \end{aligned}$$

Now, if you denote:

$$F\ln(1-p) = -\frac{1}{T}$$

Then:

$$P(t) = e^{-\frac{t}{T}}$$

Now, if $f(t)$ is the probability distribution for decay times, then:

$$P(t) = \int_{t}^{\infty} f(t')\,dt'$$

So, on solving for $f(t)$:

$$f(t) = -\frac{dP}{dt} = \frac{1}{T}e^{-\frac{t}{T}}$$

An exponential probability distribution.

To check that this is a probability distribution, it should sum to unity. To do this:

$$\int_{0}^{\infty} f(t)\,dt = \frac{1}{T}\int_{0}^{\infty} e^{-\frac{t}{T}}\,dt$$

For later convenience, note that:

$$\begin{aligned} I(l) &\equiv \int_{0}^{\infty} e^{-lt}\,dt \\ &= \left[-\frac{1}{l}e^{-lt}\right]_{0}^{\infty} \\ &= \frac{1}{l} \end{aligned}$$

So, the original integral becomes:

$$\int_{0}^{\infty} f(t)\,dt = \frac{1}{T}I(1/T) = (1/T)/(1/T) = 1$$

An average and variance can also be calculated.

Poisson Distributions:

These are used when several events might occur independently of each other, and the mean rate of occurrence is known.
For example: Geiger counter clicks in time *t*, due to background radiation; Number of earthquakes in a given time; Number of molecules in a gas in a small volume.
To do this:
The probability of no event happening, in time *t*, in the presence of a constant hazard rate a , which is its survival probability:

$$P_0(t;a) = e^{-at}$$

Use this to obtain the probability of one event happening, $P_1(t;a)$, and then extend method to $P_2, P_3, ..., P_k(t;a)$.
Note that $a\Delta t$ is both the probability of one event happening, in a short period of time, and the mean number of events in a short time – as the probability of two events occurring in a small period of time is *very* small. So, a = mean rate of occurrence.
So, to find $P_1(t;a)$:
Divide the time interval into 3 parts, assuming that the single event occurs in the short interval $[S, S+\Delta S]$. This has probability:

$$\Delta P_1 = P_0(S)\times a\Delta S \times P_0(t-(S+\Delta S)) = P_0(S).a\Delta S.P_0(t-S)$$

Where:

$P_0(S)$ is the probability of no event upto S

$a\Delta S$ is the probability of one event in ΔS

$P_0(t-S)$ is the probability of no events in remaining time $t - S$, as ΔS small

So:

$$\Delta P_1 = e^{-aS} \times a\Delta S \times e^{-a(t-S)}$$
$$= e^{-at} \times a\Delta S$$

But, the event could occur in any interval ΔS , so:

$$P_1(t;a) = \int_0^t e^{-at} a dS = e^{-at} at$$

To do this for P_{k+1}:
Divide interval into 3 parts: P_k events in $0 \le t \le S$, one event in $S \le t \le \Delta S$, and no events in $S + \Delta S \le t \le T$. This has probability:

$$\Delta P_{k+1} = P_k(S)\times a\Delta S \times P_0(t-S)$$

And, integrating as before:

$$P_{k+1}(t;a) = \int_0^t P_k(S)P_0(t-S)a\,dS$$

Now, when you start do calculate for different values of *k*, you will see that the general form is:

$$P_k(t;a) = e^{-at}\frac{(at)^k}{k!}$$

Which is known as the Poisson Distribution.
Checking that this adds up to unity, writing $l = at$:

$$\sum_{k=1}^{\infty} P_k = \sum e^{-l}\frac{l^k}{k!} = e^{-l}\left\{\sum_{k=0}^{\infty}\frac{l^k}{k!}\right\} = e^{-l}e^{l} = 1$$

Making use of the series expansion for the exponential function.
The calculations can be done similarly for the expectation value:

$$
\begin{aligned}
< k > = \sum_{k=0}^{\infty} kP_k &= \sum_{k=1}^{\infty} k \frac{l^k}{k!} e^{-l} \\
&= \sum_{k=1}^{\infty} k \frac{l^k}{k!} e^{-l} \\
&= l e^{-l} \left\{ \sum_{k=1}^{\infty} \frac{l^{k-1}}{(k-1)!} \right\} \\
&= l e^{-l} e^{l} \\
&= l
\end{aligned}
$$

$\therefore \quad < k > = l$

Also, to find the variance, do similar operations, and one will find:

$$\mathrm{var}(k) = s_k^2 = < k^2 > - < k >^2 = l$$

$\Rightarrow \quad \mathrm{var}(k) = < k > = l$

For Poisson Distributions.

Sums of Poisson random variables:

Suppose l and m are Poisson random variables, with means l and m, respectively.
For example, they could be the numbers of hurricanes and tsunamis.
So, what's the distribution for their sum $s = l + m$?
In the example, s would be the total number of disasters – hurricanes + tsunamis.
See that s may be made up in various ways:

$$(l, m) : (s,0), (s-1,1), ..., (s-k, k), ..., (0, s)$$

All mutually exclusive possibilities.
So, the probability of scenario $l = s - k$, $m = k$ is:

$$\left\{ \frac{l^{s-k}}{(s-k)!} e^{-l} \right\} \times \left\{ \frac{m^k}{k!} e^{-m} \right\}$$

Which is to be summed over the possible values for k: $k = 0,1,2,...,s$:

$$
\begin{aligned}
P_s &= \left[\sum_{k=1}^{s} \frac{l^{s-k}}{(s-k)!} \frac{m^k}{k!} s! \right] . \frac{e^{-(l+m)}}{s!} \\
&= (l + m)^s \frac{e^{-(l+m)}}{s!}
\end{aligned}
$$

Using the Binomial Theorem.
All this says, is that s is also a Poisson random variable, with mean $< s > = l + m$
To extend this to n Poisson random variables $l_1, l_2, ..., l_n$ with means $l_1, l_2, ..., l_n$, then their sum:

$$s = \sum_i l_i$$

Is a Poisson random variable with mean:

$$< s > = \sum_i l_i$$

For non-constant hazard rate $a(t)$, the Poisson distribution still holds:
The number of "counts" in the interval $[s, s + \Delta s]$ is a Poisson random variable, with mean $a(s)\Delta s$. Hence the total counts in $[0,t]$ is also a Poisson random variable, with mean:

$$l = \sum_{all\_intervals} a(s)\Delta s \rightarrow \int_0^t a(s) ds$$

For large l , Poisson distributions resemble Gaussian's. To do this, we need to look at:

$$P_n = \frac{l^n}{n!} e^{-l}$$

Near its maximum, as a function of n. That is, within a few standard deviations $\sqrt{l}$ of the mean l . To do this properly, we need an analytical approximation to $n!$. This is done using Stirling's Approximation:

Consider the integral:

$$\begin{aligned} I &= \int_1^n \ln x dx \\ &= \int_1^n 1 \ln x dx \\ &= [x \ln x]_1^n - \int_1^n x.\frac{1}{x} dx \\ &= n \ln n - n + 1 \end{aligned}$$

Now, approximate I by a series of rectangles *over* the $y = \ln x$ graph. For rectangles that terminate above the curve:

$$I < \ln 2 + \ln 3 + ... + \ln n = \ln(n!)$$

And, for rectangles with terminate just below the curve:

$$I > \ln 1 + \ln 2 + ... + \ln(n-1)$$

Now, notice:

$$\begin{aligned} \ln 1 + \ln 2 + ... + \ln(n-1) &= \ln 1 + \ln 2 + ... + \ln(n-1) + \ln n - \ln n \\ &= \ln(n!) - \ln n \end{aligned}$$

Hence:

$$I > \ln(n!) - \ln n$$

Therefore:

$$\begin{cases} \ln(n!) > I = n \ln n - n + 1 \\ \ln(n!) < I + \ln n = n \ln n - n + 1 + \ln n \end{cases}$$

Giving lower and upper bounds on $\ln(n!)$. To get an approximation to $\ln(n!)$, average the two bounds – add up, and divide by 2:

$$\ln(n!) \approx n \ln n - n + 1 + \ln\left(\sqrt{n}\right)$$

Another argument changes the +1 to $\ln(\sqrt{2p}) = 0.919$, giving Stirling's Approximation, which is:

$$\ln(n!) \approx n \ln n - n + \ln\left(\sqrt{2pn}\right)$$

The approximation improves with increasing n.

When this approximation is inserted into an expression for a Poisson distribution, use $f(n) = \ln P_n$. So:

$$\begin{aligned} f(n) &= \ln\left\{\frac{l^n}{n!} e^{-l}\right\} \\ &= n \ln l - l - \ln(n!) \\ &= n \ln l - l - \left\{n \ln n - n + \ln\left(\sqrt{2p}\right) + \tfrac{1}{2} \ln n\right\} \end{aligned}$$

Now, the max occurs at $n = \hat{n}$, with $f'(\hat{n}) = 0$:

$$f'(n) = \ln\, l - \ln n - \frac{1}{2n}$$

For $l >> 1 \Rightarrow \hat{n} \sim l$, as a first approximation, as the correction $-\frac{1}{2\hat{n}} = -\frac{1}{2l}$ is small compared with other terms.

Now, doing a Taylor expansion of $f(n)$ about $n = \hat{n}$:

$$f(n) = f(\hat{n}) + \frac{(n-\hat{n})}{1!} f'(\hat{n}) + \frac{(n-\hat{n})^2}{2!} f''(\hat{n}) + \ldots$$

But, the first derivative is zero, as a maximum, so, for a first approximation:

$$f(\hat{n}) = f(l) = -\ln\left(\sqrt{2p\, l}\right)$$

And:

$$f''(\hat{n}) = \frac{d}{dn}\left\{\ln\, l - \ln n - \frac{1}{2n}\right\}\Bigg|_{n=l}$$

$$= \left\{-\frac{1}{n} + \frac{1}{2n^2}\right\}\Bigg|_{n=l}$$

$$= -\frac{1}{l}$$

Hence, putting the two together:

$$f(n) \approx -\ln\left(\sqrt{2p\, l}\right) - \frac{(n-l)^2}{2l}$$

Remembering the initial function:

$$f(n) = \ln P_n$$

Hence:

$$P_n = \frac{e^{-\frac{(n-l)^2}{2l}}}{\sqrt{(2p\, l)}}$$

Which is the *Gaussian Limit of the Poisson Distribution*.

Binomial Distributions:

Consider an experiment of *n* independent trials, each having two possible outcomes: success or failure.
Probability of success *p*.
Probability of failure $q = 1-p$.
One possibility is that we have *k* successes, followed by *(n – k)* failures, with probability $\underbrace{p \times p \times ... \times p}_{k\_times} \times \underbrace{q \times q \times ... \times q}_{n-k\_times}$

But, the successes and failures could occur in any one of:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

So, the total probability of *k* successes will be:

$$P_k = \binom{n}{k} p^k q^{n-k}$$

Which is known as the Binomial Distribution.
To check that this is a probability distribution, it needs to add to unity; so require that:

$$\sum_{k=0}^{n} P_k = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = 1$$

Now, the Binomial distribution is actually from the binomial expansion, so:

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n$$

But, $p + q = 1$, so:

$$\sum_{k=0}^{n} P_k = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1$$

As required.
To find the expected number of successes $<k>$:

$$\begin{aligned} < k > &= \sum_{k=0}^{n} kP_k \\ &= \sum k \binom{n}{k} p^k q^{n-k} \\ &= q^n \sum \binom{n}{k} kX^k \end{aligned}$$

Where:

$$X = \frac{p}{q}$$

Notice that:

$$kX^k = X \frac{d}{dX}\left(X^k\right)$$

So:

$$< k > = q^n X \frac{d}{dX} \sum \binom{n}{k} X^k 1^{n-k}$$

$$= q^n X \frac{d}{dX}(X+1)^n$$
$$= q^n X n (X+1)^{n-1}$$
$$= q^n \frac{p}{q} n \left(\frac{p}{q}+1\right)^{n-1}$$
$$= q^{n-1} p n q^{n+1} (p+q)^{n-1}$$
$$= np$$

So:

$$< k >= np$$

For the Binomial Distribution.
To find the variance, use the same approach as in the Poisson:

$$< k(k-1) >=< k^2 > - < k >$$
$$= \left[s_k^2 + < k >^2\right] - < k >$$

So:

$$< k(k-1) >= \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k q^{n-k}$$
$$= q^n \sum \binom{n}{k} \underbrace{k(k-1) X^k}_{= X^2 \frac{d^2}{dX^2} X^k}$$
$$= q^n X^2 \frac{d^2}{dX^2} \sum \binom{n}{k} X^k$$
$$= q^n X^2 \frac{d^2}{dX^2} (X+1)^n$$
$$= q^n X^2 n(n-1)(X+1)^{n-2}$$
$$= n(n-1) p^2$$

Hence:

$$s_k^2 = n(n-1)p^2 - < k >^2 + < k >$$
$$= (n^2 p^2 - np^2) - n^2 p^2 + np$$
$$= n(p - p^2) = npq$$

So:

$$s_k^2 = npq$$

Infact, for large *n*, the Binomial Distribution can be approximated by a Gaussian Distribution, similarly to the Poisson previously. To show this:

$$P_k = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

And:

$$< k >= \bar{k} = np$$

So:

$$p = \frac{\bar{k}}{n}$$

Hence, rewriting P_k :

$$P_k = \frac{n!}{k!(n-k)!}\left(\frac{\bar{k}}{n}\right)^k \left(1-\frac{\bar{k}}{n}\right)^{n-k}$$

Now:

$$\frac{n!}{k!(n-k)!} = n(n-1)...(n-k+1) = n^k \underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{N}\right)...\left(1-\frac{k-1}{n}\right)}_{\substack{\to 1 \\ for \\ n\to\infty}}$$

Also:

$$\left(1-\frac{\bar{k}}{n}\right)^n \to e^{-\bar{k}}$$

Hence, P_k becomes:

$$P_k \approx \frac{n^k}{k!}\left(\frac{\bar{k}/n}{1-\bar{k}/n}\right)^k e^{-\bar{k}} \to \frac{\bar{k}^k}{k!} e^{-\bar{k}}$$

Which is the Poisson Distribution.