Now, the equation of a geodesic is:

$$\ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = 0$$

Where:

$$\dot{x}^i = \frac{dx^i}{ds} \qquad \text{etc}$$

Now, transform:

$$x^i = x^i(x') \qquad s = s'$$

∴

$$\dot{x}^i = \frac{\partial x^i}{\partial x'^r}\frac{\partial x'^r}{\partial s} = \frac{\partial x^i}{\partial x'^r}\dot{x}'^r$$

$$\ddot{x}^i = \frac{d}{ds}\left(\frac{\partial x^i}{\partial x'^r}\dot{x}'^r\right)$$

Notice that:

$$\frac{d}{ds}\left(\frac{\partial x^i}{\partial x'^r}\right) = \frac{\partial}{\partial x'^t}\left(\frac{\partial x^i}{\partial x'^r}\right)\frac{\partial x'^t}{\partial s}$$

Hence:

$$\ddot{x}^i = \frac{\partial x^i}{\partial x'^r}\ddot{x}'^r + \frac{\partial^2 x^i}{\partial x'^t \partial x'^r}\dot{x}'^r\dot{x}'^t$$

Substituting the above expressions for $\dot{x}^i$ and $\ddot{x}^i$ into the equation for a geodesic:

$$\frac{\partial x^i}{\partial x'^r}\ddot{x}'^r + \frac{\partial^2 x^i}{\partial x'^t \partial x'^r}\dot{x}'^r\dot{x}'^t + \Gamma^i{}_{jk}\frac{\partial x^j}{\partial x'^r}\dot{x}'^r\frac{\partial x^k}{\partial x'^t}\dot{x}'^t = 0$$

Rearranging:

$$\frac{\partial x^i}{\partial x'^r}\ddot{x}'^r + \left(\frac{\partial^2 x^i}{\partial x'^t \partial x'^r} + \Gamma^i{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}\right)\dot{x}'^r\dot{x}'^t = 0$$

Multiply both sides by $\dfrac{\partial x'^p}{\partial x^i}$. The first term is now:

$$\frac{\partial x'^p}{\partial x^i}\frac{\partial x^i}{\partial x'^r}\ddot{x}'^r$$

$$= d_r^p\ddot{x}'^r$$

$$= \ddot{x}'^p$$

Hence:

$$\ddot{x}'^p + \frac{\partial x'^p}{\partial x^i}\left(\frac{\partial^2 x^i}{\partial x'^t \partial x'^r} + \Gamma^i{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}\right)\dot{x}'^r\dot{x}'^t = 0$$

But, the equation of the geodesic in the “dashed” frame is:

$$\ddot{x}'^p + \Gamma'^p{}_{rt}\dot{x}'^r\dot{x}'^t = 0$$

Where:

$$\Gamma'^p{}_{rt} = g'^{pm}[rt,m]'$$

$$= \frac{g'^{pm}}{2}\left(\frac{\partial g'_{rm}}{\partial x'^t} + \frac{\partial g'_{tm}}{\partial x'^r} - \frac{\partial g'_{rt}}{\partial x'^m}\right)$$

Hence:

$$\Gamma'^{p}{}_{rt} = \frac{\partial x'^{p}}{\partial x^{i}}\left(\frac{\partial^2 x^i}{\partial x'^t \partial x'^r} + \Gamma^{i}{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}\right)\dot{x}'^r \dot{x}'^t \qquad (1)$$

$\therefore$

$$\Gamma'^{p}{}_{rt} = \frac{\partial x'^{p}}{\partial x^{i}}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}\Gamma^{i}{}_{jk} + \frac{\partial x'^{p}}{\partial x^{i}}\frac{\partial^2 x^i}{\partial x'^t \partial x'^r}$$

Note in the above expression, the presence of the 2nd term prevents the Γ 's transforming as tensors.

Now, multiply (1) by $\frac{\partial x^s}{\partial x'^p}$ giving:

$$\frac{\partial x^s}{\partial x'^p}\Gamma'^{p}{}_{rt} = d^{s}_{i}\left(\frac{\partial^2 x^i}{\partial x'^t \partial x'^r} + \Gamma^{i}{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}\right)$$

$$= \frac{\partial^2 x^s}{\partial x'^t \partial x'^r} + \Gamma^{s}{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t}$$

$\therefore$

$$\frac{\partial^2 x^s}{\partial x'^t \partial x'^r} = \frac{\partial x^s}{\partial x'^p}\Gamma'^{p}{}_{rt} - \Gamma^{s}{}_{jk}\frac{\partial x^j}{\partial x'^r}\frac{\partial x^k}{\partial x'^t} \qquad (2)$$

Now, the transformation of "primes" on "unprimed", sends $x \to x'$, and say that $x'' = x$, so that when you put primes on a function, unprimed become singularly primed, and singularly primed become unprimed.
Hence, (2), under this transformation, becomes:

$$\frac{\partial^2 x'^s}{\partial x^t \partial x^r} = \frac{\partial x'^s}{\partial x^p}\Gamma^{p}{}_{rt} - \Gamma'^{s}{}_{jk}\frac{\partial x'^j}{\partial x^r}\frac{\partial x'^k}{\partial x^t} \qquad (3)$$

Contravariant Differentiation:

Consider:

$$A'^i(x') = \frac{\partial x'^i}{\partial x^j}A^j(x)$$

Differentiate w.r.t. x'^k gives:

$$\frac{\partial A'^i}{\partial x'^k} = \frac{\partial}{\partial x'^k}\left(\frac{\partial x'^i}{\partial x^j}A^j\right)$$

$$= \frac{\partial x^l}{\partial x'^k}\frac{\partial}{\partial x^l}\left(\frac{\partial x'^i}{\partial x^j}A^j\right)$$

$$= \frac{\partial x^l}{\partial x'^k}\left(\frac{\partial^2 x'^i}{\partial x^l \partial x^j}A^j + \frac{\partial A^j}{\partial x^l}\frac{\partial x'^i}{\partial x^j}\right)$$

Giving:

$$\frac{\partial A'^i}{\partial x'^k} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\frac{\partial A^j}{\partial x^l} + \frac{\partial x^l}{\partial x'^k}\frac{\partial^2 x'^i}{\partial x^l \partial x^j}A^j \qquad (4)$$

Denoting:

$$B'^i{}_k = \frac{\partial A'^i}{\partial x'^k}$$

Then:

$$B'^i{}_k = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}B^j{}_l + \frac{\partial x^l}{\partial x'^k}\frac{\partial^2 x'^i}{\partial x^l \partial x^j}A^j$$

Hence $B^i{}_k$ does not transform as a tensor.
Now:

$$\frac{\partial^2 x'^i}{\partial x^l \partial x^j} = \frac{\partial x'^i}{\partial x^p}\Gamma^p{}_{lj} - \Gamma'^i{}_{pk}\frac{\partial x'^p}{\partial x^l}\frac{\partial x'^k}{\partial x^j} \qquad (5)$$

(which is the same as (3), but with symbols changed)
Substituting (5) into (4) gives:

$$\frac{\partial A'^i}{\partial x'^k} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\frac{\partial A^j}{\partial x^l} + \frac{\partial x^l}{\partial x'^k}\left(\frac{\partial x'^i}{\partial x^p}\Gamma^p{}_{lj} - \Gamma'^i{}_{pk}\frac{\partial x'^p}{\partial x^l}\frac{\partial x'^k}{\partial x^j}\right)A^j$$

$$= \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\frac{\partial A^j}{\partial x^l} + \frac{\partial x^l}{\partial x'^k}\frac{\partial x'^i}{\partial x^p}\Gamma^p{}_{lj}A^j - \Gamma'^i{}_{pk}\frac{\partial x^l}{\partial x'^k}\frac{\partial x'^p}{\partial x^l}\frac{\partial x'^k}{\partial x^k}A^j$$

The last term giving:

$$\Gamma'^i{}_{pk}\frac{\partial x^l}{\partial x'^k}\frac{\partial x'^p}{\partial x^l}\frac{\partial x'^k}{\partial x^k}A^j = \Gamma'^i{}_{pk}d^l_j\frac{\partial x'^p}{\partial x^l}A^j$$

$$= \Gamma'^i{}_{pk}\frac{\partial x'^p}{\partial x^l}A^l$$

$$= \Gamma'^i{}_{pk}A'^p$$

Hence:

$$\frac{\partial A'^i}{\partial x'^k} + \Gamma'^i{}_{pk}A'^p = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\frac{\partial A^j}{\partial x^l} + \frac{\partial x^l}{\partial x'^k}\frac{\partial x'^i}{\partial x^p}\Gamma^p{}_{lj}A^j$$

$$= \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\frac{\partial A^j}{\partial x^l} + \frac{\partial x^l}{\partial x'^k}\frac{\partial x'^i}{\partial x^j}\Gamma^j{}_{lp}A^p$$

$$= \frac{\partial x^l}{\partial x'^k}\frac{\partial x'^i}{\partial x^j}\left(\frac{\partial x^j}{\partial x^l} + \Gamma^j{}_{lp}A^p\right)$$

Putting:

$$B'^i{}_k = \frac{\partial A'^i}{\partial x'^k} + \Gamma'^i{}_{pk}A'^p$$

Hence:

$$B'^i{}_k = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}B^j{}_l$$

Therefore $B^i{}_k$ transforms as a mixed tensor.
We write:

$$B^i{}_k = A^i{}_{,k}$$

∴

$$\boxed{A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{kj}A^j}$$

This is called the covariant differentiation of the contravariant vector A^i.
Repeating for the covariant vector A_i:

$$A'_i(x') = \frac{\partial x^j}{\partial x'^i} A_j(x)$$

So, the contravariant differentiation of the covariant vector A_i is:

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^k{}_{ij} A_k$$

Some examples:

$$A^i{}_{j,k} = \frac{\partial A^i{}_j}{\partial x^k} + \Gamma^i{}_{kp} A^p{}_j - \Gamma^p{}_{jk} A^i{}_p$$

$$A^i{}_{jk,l} = \frac{\partial A^i{}_{jk}}{\partial x^l} + \Gamma^i{}_{pl} A^p{}_{jk} - \Gamma^p{}_{jl} A^i{}_{pk} - \Gamma^p{}_{kl} A^i{}_{jp}$$

$$A^{ij}{}_{k,l} = \frac{\partial A^{ij}{}_k}{\partial x^l} + \Gamma^i{}_{pl} A^{pj}{}_k + \Gamma^j{}_{pl} A^{ip} - \Gamma^p{}_{kl} A^{ij}{}_p$$

$$A^{ij}{}_{kml,t} = \frac{\partial}{\partial x^t} A^{ij}{}_{kml} + \Gamma^i{}_{pt} A^{pj}{}_{kml} + \Gamma^j{}_{pt} A^{ip}{}_{kml} - \Gamma^p{}_{kt} A^{ij}{}_{pml} - \Gamma^p{}_{mt} A^{ij}{}_{kpl} - \Gamma^p{}_{lt} A^{ij}{}_{kmp}$$

Proof that g_{ij} transforms as a covariant tensor:

$$ds' = ds$$

$$\therefore \quad g'_{ij} dx'^i dx'^j = g_{ij} dx^i dx^j$$

$$= g_{rs} dx^r dx^s$$

But:

$$dx^r = \frac{\partial x^r}{\partial x'^i} dx'^i$$

$$dx^s = \frac{\partial x^s}{\partial x'^j} dx'^j$$

$\therefore$

$$g'_{ij} dx'^i dx'^j = g_{rs} \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} dx'^i dx'^j$$

Hence, the coefficients are equal:

$$g'_{ij} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} g_{rs}$$

Hence a tensor.

So:

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^p{}_{ik} g_{pj} - \Gamma^p{}_{jk} g_{ip}$$

$$\left\{ \Gamma^p{}_{ik} = g^{pm}[ik,m] = \frac{g^{pm}}{2}\left(\frac{\partial g_{im}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m} \right) \right\}$$

$$\therefore \quad g_{pj}\Gamma^p{}_{ik} = g_{pj} g^{pm}[ik,m] = d_j^m [ik,m] = [ik,j]$$

So:

$$
\begin{aligned}
g_{ij,k} &= \frac{\partial g_{ij}}{\partial x^k} - [ik, j] - [jk, i] \\
&= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) - \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) \\
&= \frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ij}}{\partial x^k} \\
&= 0
\end{aligned}
$$

Hence, the covariant differentiation of $g_{ij,k} = 0$

Similarly:

$$g^{ij}{}_{,k} = 0$$

$$d^{i}{}_{j,k} = 0$$

$$d_{ij,k} = 0$$