Tensor Calculus:

Riemannian Space:

Suppose you have two points in normal Euclidean space:

$$P(y_1, y_2, y_3)$$
$$Q(y_1 + dy_1, y_2 + dy_2, y_3 + dy_3)$$

Now, the distance between the two points is given by:

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2$$
$$= dy_i dy_i$$

Putting:

$$y_i = y_i(x^1, x^2, x^3)$$ That is, y_i a function of x^j.

So:

$$dy_i = \frac{\partial y_i}{\partial x^j} dx^j$$

Hence:

$$ds^2 = \frac{\partial y_i}{\partial x^j}\frac{\partial y_i}{\partial x^k} dx^j dx^k$$

Putting:

$$g_{jk} = \frac{\partial y_i}{\partial x^j}\frac{\partial y_i}{\partial x^k}$$

$\therefore$ $$ds^2 = g_{jk} dx^j dx^k$$

$\therefore$ $$\boxed{ds^2 = g_{ij} dx^i dx^j} \qquad i, j = 1,2,...,n \qquad (1)$$

This is the prototype for the *metric*.

Generally, the points wont be embedded in Euclidean space.

Now, defining:

G_{ij} = cofactor of g_{ij} in the determinant $g = |g_{ij}|_n$

Then:

$$g_{ik} G_{jk} = d_{ij} g$$

$\therefore$ $$g_{ik} \frac{G_{jk}}{g} = d_{ij}$$

Define:

$$\frac{G_{jk}}{g} = g^{jk}$$

And:

$$d_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then:

$$g_{ik} g^{jk} = d_i^j$$

Also, we define:

$$g_{ij} = g_{ji} \qquad\qquad i, j = 1,2,\dots,n$$
$$g^{ij} = g^{ji}$$

Tensors:

$$P(x^1, x^2, \dots, x^n)$$
$$x^i = x^i(x'^1, x'^2, \dots, x'^n)$$
$$Q(x'^1, x'^2, \dots, x'^n)$$

Contravariant Vectors:

These transform from coordinates $(x^1, x^2, \dots, x^n)$ to coordinates $(x'^1, x'^2, \dots, x'^n)$ according to the rule:

Their components with respect to (x^i) will be denoted by A^i, and their components with respect to (x'^i) will be denoted by A'^i.

Where:

$$\boxed{A'^i = \frac{\partial x'^i}{\partial x^j} A^j} \qquad\qquad A^j(x)$$
$$A'^i(x')$$

If defined like this, then a contravariant vector.

Covariant Vectors:

These transform from x^i to x'^i according to the rule:

$$\boxed{A'_i = \frac{\partial x^j}{\partial x'^i} A_i}$$

Tensors:

Let us consider two contravariant vectors A^i & B^i, then:

$$A'^i = \frac{\partial x'^i}{\partial x^k} A^k \qquad\qquad B'^j = \frac{\partial x'^j}{\partial x^l} B^l$$

$$\therefore \qquad A'^i B'^j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^k B^l$$

Putting:

$$T'^{ij} = A'^i B'^j$$
$$T^{ij} = A^i B^j$$

Then:

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}T^{ij}$$

This forms a prototype of a second rank, contravariant tensor.
Any set of quantities T^{ij} which transform from x^i to x'^i according to this rule are called contravariant tensors of the second rank.

Similarly to covariant:

$$T'^{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}T_{kl}$$

If we take A^i and B_i, then:

$$A'^i = \frac{\partial x'^i}{\partial x^k}A^k$$

$$B'_j = \frac{\partial x^l}{\partial x'^j}B_l$$

∴

$$A'^i B'_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}A^k B_l$$

Compressing the notation:

$$T'^i{}_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}T^k{}_l$$

This forms a prototype for the rule for mixed tensors.
This one was covariant rank 1, contravariant rank 1.

$$T'^{ij}{}_k = \frac{\partial x'^i}{\partial x^p}\frac{\partial x^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}T^{pq}{}_r$$

Which is rank 2 contravariant, rank 1 covariant.
Similarly:

$$T'^i{}_{jk} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x^q}{\partial x'^j}\frac{\partial x^r}{\partial x'^k}T^p{}_{qr}$$

Raising & Lowering Indices:

$$ds^2 = g_{ij}dx^i dx^j \qquad \text{(metric)}$$

If we change from coordinates x to x', ds^2 will be an invariant:

$$ds'^2 = g'_{ij}dx'^i dx'^j$$

$$ds'^2 = ds^2$$

∴ $$g_{ij}dx^i dx^j = g'_{ij}dx'^i dx'^j$$

But:

$$x^i(x')$$

∴ $$dx^i = \frac{\partial x^i}{\partial x'^k}dx'^k$$

$$\therefore \qquad g'_{ij}dx'^i dx'^j = g_{ij}\frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}dx'^k dx'^l$$

Swapping the indices on the RHS around – *i*-*k* & *j*-*l*:

$$g'_{ij}dx'^i dx'^j = g_{kl}\frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}dx'^i dx'^j$$

Hence:

$$g'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl}$$

Hence, g_{ij} transforms as a covariant tensor of 2nd rank. That is: g_{ij} is a tensor of 2nd rank, and is covariant.

Now, $x^i = x^i(x'^i)$

We assume that we can solve these *n* equations for x' in terms of x:

$$x'^i = x'^i(x^i)$$

$$\therefore \qquad dx^i = \frac{\partial x^i}{\partial x'^j}dx'^j \qquad (2)$$

$$dx'^i = \frac{\partial x'^i}{\partial x^j}dx^j \qquad (3)$$

$$dx'^j = \frac{\partial x'^i}{\partial x^k}dx^k \qquad (4)$$

Substituting for dx'^j from (4) into (2):

$$dx^i = \frac{\partial x^i}{\partial x'^j}\frac{\partial x'^j}{\partial x^k}dx^k$$

∴

$$\frac{\partial x^i}{\partial x'^j}\frac{\partial x'^j}{\partial x^k} = d^i_k \qquad (5)$$

Substituting from (2) into (2) for dx^j:

$$dx^j = \frac{\partial x^j}{\partial x'^k}dx'^k$$

∴

$$dx'^i = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^k}dx'^k$$

∴

$$\frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^k} = d^i_k \qquad (6)$$

Consider $g_{ij}T'^{ik}$:

$$g'_{ij}T'^{ik} = \frac{\partial x^t}{\partial x'^i}\frac{\partial x^s}{\partial x'^j}g_{ts}\frac{\partial x'^i}{\partial x^m}\frac{\partial x'^k}{\partial x^l}T^{ml}$$

$$= d^t_m\frac{\partial x^s}{\partial x'^j}\frac{\partial x'^k}{\partial x^l}g_{ts}T^{ml}$$

$$= \frac{\partial x^s}{\partial x'^j}\frac{\partial x'^k}{\partial x^l}g_{ms}T^{ml}$$

Let:

$$g'_{ij}T'^{ik} = T'^{k}_{j}$$

So:

$$T'^{k}_{j} = \frac{\partial x^s}{\partial x'^j}\frac{\partial x'^k}{\partial x^l}T_s^{\ l}$$
$$= \frac{\partial x'^k}{\partial x^l}\frac{\partial x^s}{\partial x'^j}T_s^{\ l}$$

$\therefore T_j^{\ i}$ is a mixed tensor

$\therefore$

$$T_j^{\ i} = g_{kj}T^{ki}$$

Similarly, as examples:

$$g_{ik}g_{jl}T^{kl} = g_{ik}T_j^{\ k} = T^{ij}$$
$$g^{ik}g^{jl}T_{kl} = g^{ik}T^{j}{}_{k} = T^{ij}$$
$$g^{ir}g^{js}g^{kt}g_{pl}T^{pq}{}_{rst} = T_l^{\ qijk}$$
$$g_{tl}g^{ip}g^{ks}T_{pqs}{}^{t} = T^{i}{}_{q}{}^{k}{}_{l}$$
$$g_{tl}g^{ip}g^{jq}g^{ks}T_{pqs}{}^{t} = T^{ijk}{}_{l}$$

The bottom one is rank 3 contravariant, and rank 1 covariant.

Contraction:

Consider $T^{ij}{}_{k}$, making $j=k$ and sum w.r.t k. So that we are thinking of the tensor: $T^{ij}{}_{j}$.
So:

$$T'^{ij}{}_{j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}\frac{\partial x^m}{\partial x'^j}T^{kl}{}_{m}$$
$$= \frac{\partial x'^i}{\partial x^k}d_l^{\ m}T^{kl}{}_{m}$$
$$= \frac{\partial x'^i}{\partial x^k}T^{kl}{}_{l}$$

If we write,

$$T'^{i} = T'^{ij}{}_{j}$$

We have:

$$T'^{i} = \frac{\partial x'^i}{\partial x^k}T^{k}$$

Which is a contravariant tensor.

$\therefore T^{ij}{}_{j}$ is a contravariant tensor.

Similarly:

$T^{ijk}{}_{jk} = T^{i}$	$T_{ijkl}{}^{jkl} = T_i$	$T_{ijl}{}^{l} = T_{ij}$
Contravariant vector	Covariant vector	Covariant 2nd rank tensor

Consider T^{ij} contracting w.r.t $i=i$. That is the tensor T^{ii}. So:

$$T'^{ii} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^i}{\partial x^l}T^{kl}$$

Which does not simplify any more.

Geodesics:

Imagine on the surface of a sphere are two points, *p* and *q*. the shortest distance between the two is a *geodesic* – although there are two geodesics… the longest and the shortest. They are arcs of a circle.

Equations of a Geodesic:

$$l = \int_0^s ds = \int_{t_1}^{t_2} \sqrt{g_{ij}\dot{x}^i \dot{x}^j}\, dt$$

$$ds^2 = g_{ij} dx^i dx^j$$

$$= g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} dt^2$$

So:

$$ds = \sqrt{g_{ij}\dot{x}^i \dot{x}^j}\, dt$$

Now, $x^i = x^i(t)$ with $t_1 \le t \le t_2$. And $g_{ij} = g_{ij}(x(t))$.

So:

$$F = \sqrt{g_{ij}\dot{x}^i \dot{x}^j} = F(x(t), \dot{x}(t))$$

Now, from the calculus of variations:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} - \frac{\partial F}{\partial x^i} = 0 \qquad\qquad i = 1,2,\ldots,n$$

So:

$$\frac{\partial F}{\partial \dot{x}^i} = \frac{1}{2}\left(g_{ij}\dot{x}^i \dot{x}^j\right)^{-\frac{1}{2}} \frac{\partial}{\partial \dot{x}^i}\left(g_{kl}\dot{x}^k \dot{x}^l\right)$$

$$= \frac{1}{2\sqrt{g_{ij}\dot{x}^i \dot{x}^j}}\left[g_{kl}\left(d_i^k \dot{x}^l + d_i^l \dot{x}^k\right)\right]$$

Just noting that:

$$\frac{\partial}{\partial \dot{x}^i}\left(\dot{x}^k \dot{x}^l\right) = \dot{x}^l \frac{\partial \dot{x}^k}{\partial \dot{x}^i} + \dot{x}^k \frac{\partial \dot{x}^l}{\partial \dot{x}^i}$$

So, carrying on:

$$\frac{\partial F}{\partial \dot{x}^i} = \frac{1}{2\sqrt{g_{ij}\dot{x}^i \dot{x}^j}}\left(g_{il}\dot{x}^l + g_{ki}\dot{x}^k\right)$$

$$= \frac{1}{2\sqrt{g_{ij}\dot{x}^i \dot{x}^j}}\left(g_{ik}\dot{x}^k + g_{ik}\dot{x}^k\right)$$

$$= \frac{1}{\sqrt{g_{ij}\dot{x}^i \dot{x}^j}}\, g_{ik}\dot{x}^k$$

But,

$$\sqrt{g_{ij}\dot{x}^i \dot{x}^j} = \frac{ds}{dt} = \dot{s}$$

$\therefore \qquad \dfrac{\partial F}{\partial \dot{x}^i} = \dfrac{g_{ik}\dot{x}^k}{\dot{s}}$

And:

$$\frac{\partial F}{\partial x^i} = \frac{1}{2\sqrt{g_{ij}\dot{x}^i\dot{x}^j}}\dot{x}^k\dot{x}^l\frac{\partial}{\partial x^i}(g_{kl})$$

$$= \frac{\dot{x}^k\dot{x}^l}{2\dot{s}}\frac{\partial g_{kl}}{\partial x^i}$$

Hence, substituting both into the formula for the calculus of variations function:

$$\frac{d}{dt}\left(\frac{g_{ik}\dot{x}^k}{\dot{s}}\right) - \frac{1}{2\dot{s}}\frac{\partial g_{kl}}{\partial x^i}\dot{x}^k\dot{x}^l = 0$$

Now,

$$\frac{d}{dt}\left(\frac{g_{ik}\dot{x}^k}{\dot{s}}\right) = \frac{g_{ik}}{\dot{s}}\frac{\partial \dot{x}^k}{\partial t} + \dot{x}^k\frac{d}{dt}\left(\frac{g_{ik}}{\dot{s}}\right) = \frac{g_{ik}\ddot{x}^k}{\dot{s}} + \dot{x}^k\left(\frac{\dot{s}\frac{d}{dt}(g_{ik}) - g_{ik}\dot{s}}{\dot{s}^2}\right)$$

$$= \frac{g_{ik}}{\dot{s}}\ddot{x}^k + \frac{\dot{x}^k}{\dot{s}^2}\left(\dot{s}\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l - g_{ik}\ddot{s}\right)$$

$$= \frac{g_{ik}}{\dot{s}}\ddot{x}^k + \frac{\partial g_{ik}}{\partial x^l}\frac{\dot{x}^k\dot{x}^l}{\dot{s}} - \frac{g_{ik}\dot{x}^k\ddot{s}}{\dot{s}^2}$$

$\therefore$

$$\frac{g_{ik}}{\dot{s}}\ddot{x}^k + \frac{\partial g_{ik}}{\partial x^l}\frac{\dot{x}^k\dot{x}^l}{\dot{s}} - \frac{g_{ik}\dot{x}^k\ddot{s}}{\dot{s}^2} - \frac{1}{2\dot{s}}\frac{\partial g_{kl}}{\partial x^i}\dot{x}^k\dot{x}^l = 0$$

So:

$$g_{ik}\ddot{x}^k + \left(\frac{\partial g_{ik}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\right)\dot{x}^k\dot{x}^l - \frac{g_{ik}\dot{x}^k\ddot{s}}{\dot{s}} = 0$$

Now:

$$\frac{\partial g_{ik}}{\partial x^l}\dot{x}^k\dot{x}^l = \frac{\partial g_{il}}{\partial x^k}\dot{x}^l\dot{x}^k \qquad [a = b \Rightarrow a = \tfrac{1}{2}(a+b)]$$

$$= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l}\dot{x}^k\dot{x}^l + \frac{\partial g_{il}}{\partial x^k}\dot{x}^l\dot{x}^k\right)$$

Hence:

$$g_{ik}\ddot{x}^k + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right)\dot{x}^k\dot{x}^l - \frac{g_{ik}\dot{x}^k\ddot{s}}{\dot{s}} = 0$$

Notation:

$$[kl,i] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g^{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right)$$

$\therefore$

$$g_{ik}\ddot{x}^k + [kl,i]\dot{x}^k\dot{x}^l - \frac{g_{ik}\dot{x}^k\ddot{s}}{\dot{s}} = 0$$

If $s \neq 0$ we can choose $t = s$, so that $\dot{s} = 1$ & $\ddot{s} = 0$

$\therefore$

$$g_{ik}\ddot{x}^k + [kl,i]\dot{x}^k\dot{x}^l = 0$$

Multiplying through by g^{ij} :

$$g^{ij}g_{ik}\ddot{x}^k + g^{ij}[kl,i]\dot{x}^k\dot{x}^l = 0$$

But:

$$g^{ij}g_{ik} = d^i_k$$

∴

$$\ddot{x}^j + g^{ij}[kl,i]\dot{x}^k\dot{x}^l = 0$$

Notation… writing:

$$\Gamma^j{}_{kl} = g^{ij}[kl,i]$$

Then:

$$\ddot{x}^j + \Gamma^i{}_{kl}\dot{x}^k\dot{x}^l = 0$$

Or:

$$\ddot{x}^i + \Gamma^i{}_{kl}\dot{x}^k\dot{x}^l = 0$$

Where:

$$\dot{x}^k = \frac{dx^k}{ds} \;\&\; \ddot{x}^k = \frac{d^2x^k}{ds^2}$$

Now, if $s = 0$, we proceed by taking limits:

$$s = et \quad \text{with} \quad e \to 0$$

$$\therefore \quad \dot{s} = e \qquad \ddot{s} = 0$$

∴

$$g_{ik}\ddot{x}^k + [kl,i]\dot{x}^k\dot{x}^l = 0$$

Taking $e = 0 \therefore s = 0$. The equation is the same, except that:

$$\dot{x}^k = \frac{dx^k}{dt} \;\&\; \ddot{x}^k = \frac{d^2x^k}{dt^2}$$