Notes on Special & General Relativity

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Contents

1 Special Relativity

1.1 Notation, Covariant & Contravariant Transformation

Consider the neighboring events (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$ relative to an inertial frame S, where t is the time measured in S. Then, the line element (or metric) ds is given by:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \tag{1.1}$$

Where c is the speed of light in a vacuum.
Now, putting $x = x_1, y = x_2, z = x_3, ict = x_4$, then (1.1) reads:

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (d_4)^2$$

Or:

$$ds^2 = \sum_{j=1}^{4} dx_j dx_j$$

Or, using summation convention:

$$ds^2 = dx_j dx_j \tag{1.2}$$

Relative to another inertial frame S', we have:

$$ds'^2 = dx'_j dx'_j$$

In order that ds is invariant, i.e. $ds^2 = ds'^2$, it can be shown that x'_j is related to x_j by:

$$x'_j = a_{jk} x_k \tag{1.3}$$

For $j = 1, 2, 3, 4$, and taking $x'_j = 0$ when $x_j = 0$. Where a_{jk} are constants satisfying the following equations:

$$a_{jl} a_{kl} = a_{lj} a_{lk} = \delta_{jk} \tag{1.4}$$

Using (1.3) and (1.4), we can also write:

$$x_j = a_{kj} x'_k \tag{1.5}$$

Let A^j be the components of a contravariant vector. Then:

$$A'^j = \frac{\partial x'^j}{\partial x^k} A^k = \frac{\partial x'_j}{\partial x_k} A^k$$

And, using (1.3) gives:

$$A'^j = a_{jk} A^k \tag{1.6}$$

Where we have literally just divided line elements, to leave the constants.
If A_j are the components of some covariant vector, then:

$$A'_j = \frac{\partial x^k}{\partial x'^j} A_k = \frac{\partial x_k}{\partial x'_j} A_k$$

Then, via (1.5), we have:

$$A'_j = a_{jk} A_k \tag{1.7}$$

Thus, in our notation of special relativity, contravariant and covariant vectors are transformed by the same laws. That is, contravariant and covariant vectors are indistinguishable. Similar remarks apply to tensors. There is another notation for special relativity in which they are distinguishable. From now on, we will denote vectors (and tensors) with subscripts.

1.2 Proper Time

Putting $ds^2 = -c^2 d\tau^2$ in (1.1) gives:

$$d\tau^2 = \left(1 - \frac{1}{c^2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}\right) dt^2$$

This is done via trvial rearrangement. We write this as:

$$dt = \beta d\tau \tag{1.8}$$

Where:

$$\beta \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where, v is the speed:

$$\begin{aligned} v &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{\dot{x}_k \dot{x}_k} \end{aligned}$$

Where:

$$\dot{x}_k = \frac{dx_k}{dt}$$

From (1.8), we see that:

$$\tau = \int_{t_0}^{t} \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

We call τ the *proper time*.

1.3 Motion of a Particle

Let a particle P be in motion relative to an inertial frame S. Suppose that at time t the coordinates of P are (x_1, x_2, x_3), then the velocity $\boldsymbol{v} = (v_1, v_2, v_3)$ of P at this instant are defined as:

$$\boldsymbol{v} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right)$$

That, is, the components are given by:

$$v_k = \frac{dx_k}{dt} = \dot{x}_k \tag{1.9}$$

If $\boldsymbol{f} = (f_1, f_2, f_3)$ is the resultant force acting on P, by Newton's 2nd law we have:

$$f_k = \frac{d}{dt}(mv_k) \tag{1.10}$$

Where m is the mass of P, measured in S. In special relativity, m is not a constant, but is given by:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta m_0$$

Where m_0 is the mass of P as measured in the rest frame of P; i.e. the frame in which P is at rest. Thus:

$$m = \beta m_0 \tag{1.11}$$

1.4 4-Velocity & 4-Force

I shall try to be consistent with notating the difference in 3- and 4-vectors. Suppose we have the 4-vector for force, I shall denote the whole as F, which will have 3-spatial, and 1-temporal components. The spatial part I shall denote $\boldsymbol{f}$ (similarly, the spatial part of 4-velocity u shall be the usual $\boldsymbol{v}$). Notice the difference in font between u and $\boldsymbol{u}$. This is the difference that will be used.

1.4.1 4-Velocity

The 4-velocity $\mathsf{u} = (u_1, u_2, u_3, u_4)$ is defined by:

$$u_j = \frac{dx_j}{d\tau} \tag{1.12}$$

For $j = 1, 2, 3, 4$. We can write (1.12) as:

$$\begin{aligned} u_j &= \frac{dx_j}{dt}\frac{dt}{d\tau} \\ &= \beta\frac{dx_j}{dt} \end{aligned}$$

By using (1.8). Hence, the spatial and temporal components:

$$\begin{aligned} u_i &= \beta\frac{dx_i}{dt} \qquad i = 1, 2, 3 \\ u_4 &= \beta\frac{dx_4}{dt} \\ &= \beta\frac{d}{dt}(ict) \\ &= ic\beta \end{aligned}$$

Therefore, the 4-velocity u is given by:

$$\mathsf{u} = \beta(\boldsymbol{v}, ic) \tag{1.13}$$

Where $\boldsymbol{v} = (v_1, v_2, v_3)$, the normal 3-velocity.

1.4.2 The 4-Force

The 4-force is defined by $\mathsf{F} = (F_1, F_2, F_3, F_4)$:

$$F_j = m_0 \frac{du_j}{d\tau} \tag{1.14}$$

From (1.2), $ds^2 = -c^2 d\tau^2 = dx_j dx_j$. Which is just:

$$\frac{dx_j}{d\tau}\frac{dx_j}{d\tau} = -c^2$$

That is, upon comparison with the definition of the 4-velocity:

$$u_j u_j = -c^2$$

Therefore, taking the τ-derivative, noting that c^2 is a constant:

$$\frac{d}{d\tau}(u_j u_j) = 0$$

And a trivial application of the product rule:

$$\begin{aligned} 2\frac{du_j}{d\tau}u_j &= 0 \\ \Rightarrow \frac{du_j}{d\tau}u_j &= 0 \\ \Rightarrow u_j m_0 \frac{du_j}{d\tau} &= 0 \end{aligned}$$

Where we see that we have the 4-force. Hence:

$$u_j F_j = 0 \tag{1.15}$$

But, from (1.13) and (1.14):

$$\begin{aligned} \mathsf{F} &= m_0 \frac{d\mathsf{u}}{d\tau} \\ &= m_0 \frac{d\mathsf{u}}{dt}\frac{dt}{d\tau} \\ &= m_0 \beta \frac{d}{dt}(\beta \boldsymbol{v}, i\beta c) \\ &= \beta \left(\frac{d}{dt} m_0 \beta \boldsymbol{v}, ic\frac{d}{dt} m_0 \beta \right) \end{aligned}$$

Where we have noticed that $\frac{dt}{d\tau} = \beta$, and that $\beta m_0 = m$ (which we have previously shown). Hence, this becomes:

$$\mathsf{F} = \beta\left(\frac{d}{dt}m\boldsymbol{v}, ic\frac{d}{dt}m\right)$$

And, we know that $\boldsymbol{f} = \frac{d}{dt}m\boldsymbol{v}$, the standard 3-force. Hence, the 4-force is:

$$\mathsf{F} = \beta(\boldsymbol{f}, ic\dot{m}) \tag{1.16}$$

Now, substituting (1.13) and (1.16) into (1.15), which is basically finding the scalar product of the 4-velocity and force, and putting it equal to zero (the previously found statment that $u_jF_j = 0$). We thus get:

$$\boldsymbol{f}\cdot\boldsymbol{v} - c^2\dot{m} = 0$$

Hence, upon rearrangement:

$$\dot{m} = \frac{\boldsymbol{f}\cdot\boldsymbol{v}}{c^2} \tag{1.17}$$

From (1.17) we see that the 4-force (1.16) can be written:

$$\mathsf{F} = \beta\left(\boldsymbol{f}, \frac{i\boldsymbol{f}\cdot\boldsymbol{v}}{c}\right) \tag{1.18}$$

Now, let $d\omega_0$ be the volume measured in the rest frame of P, then if $d\omega$ is the volume measured in S, then:

$$d\omega = d\omega_0\sqrt{1-\frac{v^2}{c^2}}$$

Where v is the speed of P, $v = \sqrt{v_kv_k}$. Thus, we see:

$$d\omega_0 = \beta d\omega \tag{1.19}$$

Now, defining:

$$\mathsf{D} \equiv \frac{\mathsf{F}}{d\omega_0}$$

We see that:

$$\begin{aligned}\mathsf{D} &= \frac{\mathsf{F}}{d\omega_0}\\ &= \frac{\mathsf{F}}{\beta d\omega}\\ &= \left(\frac{\boldsymbol{f}}{d\omega}, \frac{i\boldsymbol{f}\cdot\boldsymbol{v}}{cd\omega}\right)\end{aligned}$$

Now, let the force per unit volume be $\boldsymbol{d} \equiv \frac{\boldsymbol{f}}{d\omega}$, then the above becomes:

$$\mathsf{D} = \left(\boldsymbol{d}, \frac{i\boldsymbol{d}\cdot\boldsymbol{v}}{c}\right) \tag{1.20}$$

It can be shown that F, D transform as tensors.

1.5 Motion of a Perfect Fluid

It can be shown that in the case of a perfect fluid, that the components of $\mathsf{D} = (D_1, D_2, D_3, D_4)$ can be written:

$$D_j = \frac{\partial T_{jk}}{\partial x_k} \qquad j = 1, 2, 3, 4 \tag{1.21}$$

Where T_{jk} are the components of a tensor known as the energy momentum tensor:

$$T_{jk} = \left(\rho_0 + \frac{p}{c^2}\right) u_j u_k + p\delta_{jk} \tag{1.22}$$

Where ρ_0 is the density (mass per unit volume) measured in the instantaneous rest frame of a typical particle of the fluid. p is the pressure (force per unit area). p is an invariant, i.e. $p' = p$, for any two inertial frames S and S'.
The derivation of T_{jk} can be found in *Introduction to Tensor Calculus, Relativity and Cosmology*, D.F.Lawden, 3rd Ed, pp50-57.

2 General Relativity

2.1 Tensor Analysis

Throughout, super-scripts and sub-scripts will be deonoted by $i, j, k, l, \ldots$, these are not limited to being $1, 2, 3, 4$; but will range to an arbitrary n. There may be places where $i = \sqrt{-1}$ and the sub-script i will be confused, but context will sort out the confusion.

2.1.1 Geodesic Coordinates

Consider a change of coordinates $x^i \to x'^i$, as defined by:

$$x^i = x'^i + \frac{1}{2} a^i_{\ jk} x'^j x'^k \tag{2.1}$$

Where:

$$a^i_{\ jk} = a^i_{\ kj} \tag{2.2}$$

And are constants. We note that if $x'^i = 0$, then $x^i = 0$. Now, let us differentiate the above expression, with respect to x'^r. That is:

$$\begin{aligned}
\frac{\partial x^i}{\partial x'^r} &= \frac{\partial x'^i}{\partial x'^r} + \frac{\partial}{\partial x'^r}\frac{1}{2}a^i_{\ jk}x'^j x'^k \\
&= \delta^i_r + \frac{1}{2}\left(\frac{\partial}{\partial x'^r}x'^j x'^k\right)a^i_{\ jk} \\
&= \delta^i_r + \frac{1}{2}\left(x'^k\delta^j_r + x'^j\delta^k_r\right)a^i_{\ jk} \\
&= \delta^i_r + \frac{1}{2}\left(x'^k\delta^j_r a^i_{\ jk} + x'^j\delta^k_r a^i_{\ jk}\right) \\
&= \delta^i_r + \frac{1}{2}\left(x'^k a^i_{\ rk} + x'^j a^i_{\ jr}\right) \\
&= \delta^i_r + \frac{1}{2}\left(x'^k a^i_{\ rk} + x'^k a^i_{\ kr}\right) \\
&= \delta^i_r + \frac{1}{2}\left(x'^k a^i_{\ rk} + x'^k a^i_{\ rk}\right) \\
&= \delta^i_r + a^i_{\ rk}x'^k \\
&= \delta^i_r + a^i_{\ rj}x'^j
\end{aligned}$$

In this derivation, we have make extensive use of the Kronecker delta: used in noting that when differentiating a component with respect to another, gives zero; and with respect to itself gives one. We have also used the symmetry in (2.2). We have also used the redundency of repeated indices in switching between using other letter, so long as order & context is preserved. This can be seen by inspecting everything above carefully.
Differentiating again:

$$\begin{aligned}
\frac{\partial^2 x^i}{\partial x'^s \partial x'^r} &= \frac{\partial}{\partial x'^s}\left(\delta^i_r + a^i_{\ rj}x'^j\right) \\
&= a^i_{\ rj}\delta^j_s \\
&= a^i_{\ rs}
\end{aligned}$$

So, we have derived:

$$\begin{aligned}
\frac{\partial x^i}{\partial x'^r} &= \delta^i_r + a^i_{\ rj}x'^j \\
\frac{\partial^2 x^i}{\partial x'^s \partial x'^r} &= a^i_{\ rs}
\end{aligned}$$

These will be used later.
In these two expressions, if we put $x'^i = 0$, then we get:

$$\left.\frac{\partial x^i}{\partial x'^r}\right)_{x'^i=0} = \delta^i_r \tag{2.3}$$

$$\left.\frac{\partial^2 x^i}{\partial x'^s \partial x'^r}\right)_{x'^i=0} = a^i_{\ rs} \tag{2.4}$$

Now, we transform the metric (tensor) via:

$$g'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl} \tag{2.5}$$

Differentiating:

$$\begin{aligned}\frac{\partial g'_{ij}}{\partial x'^r} &= \frac{\partial}{\partial x'^r}\left(\frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl}\right) \\ &= \left(\frac{\partial x^l}{\partial x'^j}\frac{\partial^2 x^k}{\partial x'^r \partial x'^i} + \frac{\partial x^k}{\partial x'^i}\frac{\partial^2 x^l}{\partial x'^r \partial x'^j}\right) g_{kl} + \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}\frac{\partial x^t}{\partial x'^r}\frac{\partial g_{kl}}{\partial x^t}\end{aligned}$$

Where we have just used the product rule. the expression to the right has just been multiplied by $\frac{\partial x^t}{\partial x^t}$, i.e. not essentially doing anything.
Now, if we compare the above with the derived expressions (2.3) and (2.3), and putting $x'i = 0$, we get:

$$\begin{aligned}\left.\frac{\partial g'_{ij}}{\partial x'^r}\right)_{x'^i=0} &= \left(\delta^l_j a^k_{\ ir} + \delta^k_i a^l_{\ jr}\right) g_{kl} + \delta^k_i \delta^l_j \delta^t_r \frac{\partial g_{kl}}{\partial x^t} \\ &= a^k_{\ ir} g_{kj} + a^l_{\ jr} g_{il} + \frac{\partial g_{ij}}{\partial x^r}\end{aligned}$$

Now, writing

$$a^k_{\ ir} g_{kj} = a_{irj} \tag{2.6}$$

We can write the above as:

$$\left.\frac{\partial g'_{ij}}{\partial x'^r}\right)_{x'^i=0} = a_{irj} + a_{jri} + \frac{\partial g_{ij}}{\partial x^r} \tag{2.7}$$

Now, by our previously stated symmetry $a^i_{\ jk} = a^i_{\ kj}$, we see that:

$$\begin{aligned} a_{rij} &= a^k_{\ ri} g_{kj} \\ &= a^k_{\ ir} g_{kj} \\ &= a_{irj}\end{aligned}$$

After using the relation (2.6). Hence:

$$a_{rij} = a_{irj} \tag{2.8}$$

Now, by making the cyclic interchange of indices $i \to r \to j \to i$, we can write (2.7) as (if we do the change twice):

$$\left.\frac{\partial g'_{ri}}{\partial x'^j}\right)_{x'^i=0} = a_{rji} + a_{ijr} + \frac{\partial g_{ri}}{\partial x^j} \tag{2.9}$$

$$\left.\frac{\partial g'_{jr}}{\partial x'^i}\right)_{x'^i=0} = a_{jir} + a_{rij} + \frac{\partial g_{jr}}{\partial x^i} \tag{2.10}$$

Hence, adding (2.7) and (2.9), making use of (2.8):

$$\left.\frac{\partial g'_{ij}}{\partial x'^r}\right)_{x'^i=0} + \left.\frac{\partial g'_{ri}}{\partial x'^j}\right)_{x'^i=0} = 2a_{rji} + a_{rij} + a_{jir} + \frac{\partial g_{ij}}{\partial x^r} + \frac{\partial g_{ri}}{\partial x^j} \tag{2.11}$$

Subtracting (2.10) from (2.11) gives:

$$\left.\frac{\partial g'_{ij}}{\partial x'^r}\right)_{x'^i=0} + \left.\frac{\partial g'_{ri}}{\partial x'^j}\right)_{x'^i=0} - \left.\frac{\partial g'_{jr}}{\partial x'^i}\right)_{x'^i=0} = 2a_{rji} + a_{rij} + a_{jir} + \frac{\partial g_{ij}}{\partial x^r} + \frac{\partial g_{ri}}{\partial x^j} - a_{jir} - a_{rij} - \frac{\partial g_{jr}}{\partial x^i}$$

I shall now leave out the indication that everything is evaluated at $x'^i = 0$. So, this results in:

$$\frac{\partial g'_{ij}}{\partial x'^r} + \frac{\partial g'_{ri}}{\partial x'^j} - \frac{\partial g'_{jr}}{\partial x'^i} = 2a_{rji} + \frac{\partial g_{ij}}{\partial x^r} + \frac{\partial g_{ri}}{\partial x^j} - \frac{\partial g_{jr}}{\partial x^i}$$

Now, we shall make use of some notation:

$$[jr, i] \equiv \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^r} + \frac{\partial g_{ri}}{\partial x^j} - \frac{\partial g_{jr}}{\partial x^i}\right)$$

Hence, we have:

$$[jr, i]' = a_{rji} + [jr, i] \tag{2.12}$$

Now, if we choose:

$$a_{rji} = [jr, i]' \tag{2.13}$$

Then:

$$[jr, i] = 0 \tag{2.14}$$

From (2.6) and (2.13), we have

$$a^k_{rj} g_{ki} = [jr, i]' \tag{2.15}$$

But, from (2.5), putting $x'^i = 0$ (and therefore $x^i = 0$), and using (2.3) simply gives:

$$g'_{ij} = \delta^k_i \delta^l_j g_{kl} = g_{ij}$$

Hence, (2.15) reads:

$$a^k_{rj} g'_{ki} = [jr, i]'$$

Hence:

$$a^k_{rj} = g'^{ki}[jr, i]'$$

Or, using some notation:

$$a^k_{rj} = \Gamma'^k_{jr}$$

So, we could also write (2.14) as:

$$g^{ki}[jr, i] = 0$$

That is:

$$\Gamma^k_{jr} = 0 \tag{2.16}$$

Now, let P be any point having coordinates $(a^1, a^2, \ldots, a^n)$ in x, and $(a'^1, a'^2, \ldots, a'^n)$ in x'. Replacing x^i by $x^i - a^i$ and x'^i by $x'^i - a'^i$ in (2.1) gives:

$$x^i - a^i = x'^i - a'^i + \frac{1}{2}a^i_{kj}(x'^j - a'^j)(x'^k - a'^k)$$

From which it follows that when $x'^i = a'^i$, $x^i = a^i$, (2.16) gives:

$$(\Gamma^k_{jr})_P = (\Gamma^k_{jr})_{x^i=a^i} = 0 \tag{2.17}$$

Thus, at any point we can use geodesic coordinates, such that the components of all the Γ's are zero at that point.

2.2 The Riemann-Christoffel Tensor

Let A_i be the components of a covariant tensor. Then, the covariant derivative $A_{i,j}$ of A_i is given by:

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ji} A_k \tag{2.18}$$

Where this was shown in the previous set of Tensor Calculus notes, and we also see that $A_{i,j}$ is a tensor of rank 2. Also:

$$\begin{aligned} A_{i,jk} &= (A_{i,j})_k && (2.19)\\ &= \frac{\partial A_{i,j}}{\partial x^k} - \Gamma^r_{ki} A_{r,j} - \Gamma^s_{kj} A_{s,i} && (2.20) \end{aligned}$$

Which, upon insertion of (2.18) into (2.21) gives:

$$\begin{aligned} A_{i,jk} &= \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - \Gamma^l_{ji} A_l\right) - \Gamma^r_{ki}\left(\frac{\partial A_r}{\partial x^j} - \Gamma^t_{jr} A_t\right) - \Gamma^s_{kj}\left(\frac{\partial A_i}{\partial x^s} - \Gamma^t_{is} A_t\right)\\ &= \frac{\partial^2 A_i}{\partial x^k \partial x^j} - A_l \frac{\partial \Gamma^l_{ji}}{\partial x^k} - \Gamma^l_{ji}\frac{\partial A_l}{\partial x^k} - \Gamma^r_{ki}\frac{\partial A_r}{\partial x^j} + \Gamma^r_{ki}\Gamma^t_{jr} A_t - \Gamma^s_{kj}\frac{\partial A_i}{\partial x^s} + \Gamma^s_{kj}\Gamma^t_{is} A_t \end{aligned}$$

Where we have just expanded out all the brackets, using product rule where applicable. Rearranging, and renaming some indices:

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \left(\Gamma^s_{ij}\frac{\partial A_s}{\partial x^k} + \Gamma^s_{ki}\frac{\partial A_s}{\partial x^j} + \Gamma^s_{kj}\frac{\partial A_i}{\partial x^s}\right) + A_t\left(\Gamma^s_{kj}\Gamma^t_{is} + \Gamma^r_{ki}\Gamma^t_{jr} - \frac{\partial \Gamma^t_{ij}}{\partial x^k}\right)$$

Hence:

$$A_{i,kj} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \left(\Gamma^s_{ik}\frac{\partial A_s}{\partial x^j} + \Gamma^s_{ji}\frac{\partial A_s}{\partial x^k} + \Gamma^s_{jk}\frac{\partial A_i}{\partial x^s}\right) + A_t\left(\Gamma^s_{jk}\Gamma^t_{is} + \Gamma^r_{ji}\Gamma^t_{kr} - \frac{\partial \Gamma^t_{ik}}{\partial x^j}\right)$$

Now, we note the following things:

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} \qquad \Gamma^s_{jk} = \Gamma^s_{kj}$$

So:

$$A_{i,jk} - A_{i,kj} = \left(\Gamma^r_{ki}\Gamma^t_{jr} - \Gamma^r_{ji}\Gamma^t_{kr} + \frac{\partial \Gamma^t_{ik}}{\partial x^j} - \frac{\partial \Gamma^t_{ij}}{\partial x^k}\right) A_t$$

So:

$$A_{i,jk} - A_{i,kj} = R^t_{\ ijk} A_t \tag{2.21}$$

Where:

$$R^t_{\ ijk} = \Gamma^r_{ki}\Gamma^t_{jr} - \Gamma^r_{ji}\Gamma^t_{kr} + \frac{\partial \Gamma^t_{ik}}{\partial x^j} - \frac{\partial \Gamma^t_{ij}}{\partial x^k} \tag{2.22}$$

We now prove that R^t_{ijk} is a tensor.
Changing the coordinates from x^i to x'^i gives:

$$A'_{i,jk} - A'_{i,kj} = R'^t_{ijk} A'_t \tag{2.23}$$

But, on the LHS, $A'_{i,kj}$ is a tensor. Thus, the LHS transforms as:

$$A'_{i,jk} - A'_{i,kj} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} (A_{r,sl} - A_{r,ls})$$

So, from (2.21) and (2.23), this gives:

$$A'_t R'^t_{ijk} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} R^t_{rsl} A_t$$

But, the A_t are just components of a covariant vector, which transform as:

$$A'_t = \frac{\partial x^a}{\partial x'^t} A_a$$

Hence:

$$\frac{\partial x^a}{\partial x'^t} A_a R'^t_{ijk} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} R^t_{rsl} A_t$$

But, the A_t are arbitrary. Choose $A_t = \delta_{tm}$, $A_a = \delta_{am}$. Hence:

$$\frac{\partial x^m}{\partial x'^t} R'^t_{ijk} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} R^m_{rsl}$$

Hence:

$$R'^t_{ijk} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} \frac{\partial x'^t}{\partial x^m} R^m_{rsl}$$

Therefore, R^i_{jkl} are the components of a fourth rank tensor. It is called the *Riemann-Christoffel tensor*

2.3 The Ricci Tensor

Contracting R^t_{jkl} with respect to t and l gives:

$$R^t_{jkt} = R_{jk} \tag{2.24}$$

Where R_{jk} is known as the Ricci tensor.

Now, $R_{ijkl} = g_{it} R^t_{jkl}$

2.3.1 Properties of R_{ijkl}

Now:

$$\begin{aligned} R_{ijkl} &= g_{it} R^t_{jkl} \\ &= g_{it} \left(\Gamma^t_{kr} \Gamma^r_{jl} - \Gamma^t_{lr} \Gamma^r_{jk} + \frac{\partial \Gamma^t_{jl}}{\partial x^k} - \frac{\partial \Gamma^t_{jk}}{\partial x^l} \right) \\ &= \frac{\partial}{\partial x^k} \left(g_{it} \Gamma^t_{jl} \right) - \frac{\partial}{\partial x^l} \left(g_{it} \Gamma^t_{jk} \right) - \Gamma^t_{jl} \frac{\partial g_{it}}{\partial x^k} + \Gamma^t_{jk} \frac{\partial g_{it}}{\partial x^l} + g_{it} \left(\Gamma^t_{kr} \Gamma^r_{jl} - \Gamma^t_{lr} \Gamma^r_{jk} \right) \end{aligned}$$

Where the first step used the definition (2.22), and the second effectively did nothing, except arrange things in a different fashion, just subtracting terms that are added by the differentiation.

Now, choosing geodesic coordinates at any arbitrary point 0, we can make:

$$\Gamma^i_{jk} = 0$$

As we have previously shown. Hence:

$$R_{ijkl} = \frac{\partial}{\partial x^k}\left(g_{it}\Gamma^t_{jl}\right) - \frac{\partial}{\partial x^l}\left(g_{it}\Gamma^t_{jk}\right)$$

But:

$$\begin{aligned} g_{it}\Gamma^t_{jl} &= [jl,i] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^i}\right) \\ g_{it}\Gamma^t_{jk} &= [jk,i] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) \end{aligned}$$

Hence:

$$\begin{aligned} \frac{\partial}{\partial x^k}\left(g_{it}\Gamma^t_{jl}\right) &= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{jl}}{\partial x^k \partial x^i}\right) \\ \frac{\partial}{\partial x^l}\left(g_{it}\Gamma^t_{jk}\right) &= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^l \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^l \partial x^i}\right) \end{aligned}$$

Therefore:

$$\frac{\partial}{\partial x^k}\left(g_{it}\Gamma^t_{jl}\right) - \frac{\partial}{\partial x^l}\left(g_{it}\Gamma^t_{jk}\right) = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^l \partial x^i} - \frac{\partial^2 g_{jl}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j}\right)$$

Where we have of course noted that $\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 g_{ij}}{\partial x^k \partial x^k}$. Therefore, we have that:

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{jk}}{\partial x^l \partial x^i} - \frac{\partial^2 g_{jl}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j}\right)$$

Now, let us inspect what happens if we swap various indices:

$$\begin{aligned} R_{jikl} &= \frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x^k \partial x^i} + \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^l \partial x^i}\right) \\ &= -R_{ijkl} \end{aligned}$$

Similarly, we see that:

$$R_{ijlk} = -R_{ijkl}$$

And also:

$$R_{klij} = R_{ijkl}$$

This last one uses the fact that $g_{ij} = g_{ji}$.
Remember, all this has been for the 0 point. Thus, as 0 is arbitrary, we have:

$$\begin{aligned} R_{ijkl} &= -R_{jikl} && (2.25) \\ R_{ijkl} &= -R_{ijlk} && (2.26) \\ R_{ijkl} &= R_{klij} && (2.27) \end{aligned}$$

Again, using our condition that $\Gamma^i_{jk} = 0$, and (2.22), we have that:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l}$$

Cyclic interchange of indices $j \to k \to l \to j$ gives:

$$R^i_{klj} = \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^j}$$

And again:

$$R^i_{ljk} = \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k}$$

Adding:

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$$

Where we have used that $\Gamma^i_{jk} = \Gamma^i_{kj}$. Also, this has been for the 0 point; which is, however, arbitrary, so:

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \tag{2.28}$$

Thus, the R^j_{jkl} are not independant, and must always satisfy (2.25)-(2.28).

2.4 Bianchi's Identities

Let us differentiate:

$$R^i_{jkl,m} = \frac{\partial}{\partial x^m} R^i_{jkl}$$

Again, by noting that $\Gamma^i_{jk} = 0$ at position 0; we use (2.22):

$$R^i_{jkl,m} = \frac{\partial^2 \Gamma^i_{jl}}{\partial x^m \partial x^k} - \frac{\partial^2 \Gamma^i_{jk}}{\partial x^m \partial x^l}$$

Cyclic interchange of indices $k \to l \to m \to k$ gives:

$$R^i_{jlm,k} = \frac{\partial^2 \Gamma^i_{jm}}{\partial x^k \partial x^l} - \frac{\partial^2 \Gamma^i_{jl}}{\partial x^k \partial x^m}$$

And cycling again:

$$R^i_{jmk,l} = \frac{\partial^2 \Gamma^i_{jk}}{\partial x^l \partial x^m} - \frac{\partial^2 \Gamma^i_{jm}}{\partial x^l \partial x^k}$$

Adding gives:

$$R^i_{jkl,m} + R^i_{jlm,k} + R^i_{jmk,l} = 0$$

Again, the point 0 used above is arbitrary, so we have:

$$R^i_{jkl,m} + R^i_{jlm,k} + R^i_{jmk,l} = 0 \tag{2.29}$$

Similarly, by lowering the i:

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0 \tag{2.30}$$

(2.29) and (2.30) are known as *Bianchi's indentities.*
Let us use (2.30) on $R_{srjt,i}$:

$$R_{srjt,i} + R_{srti,j} + R_{srij,t} = 0$$

Multiplying by $g^{ir}g^{ts}$ gives:

$$(g^{ir}g^{ts}R_{srjt,i}) + (g^{ir}g^{ts}R_{srti,j}) + (g^{ir}g^{ts}R_{srij,t}) = 0$$

Now, we also know the following:

$$\begin{aligned}
g^{ir}g^{ts}R_{srjt} &= g^{ir}R^{t}_{\ rjt} \\
&= g^{ir}R_{rj} \\
&= R^{i}_{j} \\
g^{ir}g^{ts}R_{srti} &= -g^{ir}g^{ts}R_{rsti} \\
&= -g^{ts}R^{i}_{\ sti} \\
&= -g^{ts}R_{st} \\
&= -R^{s}_{s} \\
&= -R \\
g^{ir}g^{ts}R_{srij} &= g^{ir}g^{ts}R_{rsji} \\
&= g^{ts}R^{i}_{\ sji} \\
&= g^{ts}R_{sj} \\
&= R^{t}_{j}
\end{aligned}$$

Hence, putting all this together:

$$(R^{i}_{j})_{,i} - R_{,j} + (R^{t}_{j})_{,t} = 0$$

We notice that the last and first terms are identical. Hence:

$$2R^{i}_{j,i} - R_{,j} = 0$$

Which is the same as:

$$R^{i}_{j,i} - \tfrac{1}{2}\delta^{i}_{j}R_{,i} = 0$$

That is:

$$(R^{i}_{j} - \tfrac{1}{2}\delta^{i}_{j}R)_{,i} = 0 \tag{2.31}$$

3 Einstein's General Relativity

If a person jumps from the roof of a building, then relative to the person, there is no gravity; thus, he/she can set up a frame of reference $S'(x')$ which is inertial. $S'(x')$ is limited in space and times, ideally being infinitesimal. $S'(x')$ is called the free-fall frame. In the frame, special relativity applies. The metric in $S'(x')$ is $ds'^2 = dx'_i dx'_i = \delta_{ij} dx'_i dx'_j$. Thus:

$$g'_{ij} = \delta_{ij} \tag{3.1}$$

If $S(x)$ is the general frame set up in the gravitational field, then $S(x)$ is not inertial. But, $x^i = x^i(x')$ (that is, a function of (x'_1, x'_2, x'_3, x'_4)); so that in $S(x)$, space and times lose their objective meaning.

Let T'_{jk} be the energy-momentum tensor for a perfect fluid. Then, if $D'_j = 0$, we have from (1.21) that:

$$\frac{\partial T'_{jk}}{\partial x'^k} = 0 \tag{3.2}$$

Depending on the distribution of matter and energy, the form of T'_{jk} will not necessarily be given by (1.21), but in all cases, T'_{jk} is assumed to satisfy (3.2); and the condition that:

$$T'_{jk} = T'_{kj}$$

Let T_{jk} be the components of the energy-momentum tensor in $S(x)$; then:

$$T_{jk} = \frac{\partial x'^r}{\partial x^j} \frac{\partial x'^s}{\partial x^k} T'_{rs} \tag{3.3}$$

If in $S(x)$, $ds^2 = g_{ij} dx^i dx^j$; then:

$$g_{ij} = \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} g'_{rs} = \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} \delta_{rs} \tag{3.4}$$

That is:

$$g_{ij} = \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^r}{\partial x^j} \tag{3.5}$$

Similarly:

$$g^{ij} = \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^r} \tag{3.6}$$

Now:

$$\begin{aligned} T^i_j &= g^{ik} T_{kj} \\ &= \frac{\partial x^i}{\partial x'^r} \frac{\partial x^k}{\partial x'^r} T_{kj} \\ &= \frac{\partial x^i}{\partial x'^r} \frac{\partial x^k}{\partial x'^r} \frac{\partial x'^s}{\partial x^k} \frac{\partial x'^t}{\partial x^j} T'_{st} \end{aligned}$$

But:

$$\frac{\partial x^k}{\partial x'^r} \frac{\partial x'^s}{\partial x^k} = \delta^s_r$$

So:

$$\begin{aligned} T^i_j &= \frac{\partial x^i}{\partial x'^r} \delta^s_r \frac{\partial x'^t}{\partial x^j} T'_{st} \\ &= \frac{\partial x^i}{\partial x'^s} \frac{\partial x'^t}{\partial x^j} T'_{st} \end{aligned}$$

Since $T^i_{j,i}$ is a tensor; thus:

$$T^i_{j,i} = \frac{\partial x^i}{\partial x'^r}\frac{\partial x'^s}{\partial x^j}\frac{\partial x'^t}{\partial x^i}T'^r_{s,t}$$

But:

$$\frac{\partial x^i}{\partial x'^r}\frac{\partial x'^t}{\partial x^i} = \delta^t_r$$

Thus:

$$\begin{aligned} T^i_{j,i} &= \frac{\partial x'^s}{\partial x^j}\delta^t_r T'^r_{s,t} \\ &= \frac{\partial x'^s}{\partial x^j}T'^r_{s,r} \\ &= \frac{\partial x'^s}{\partial x^j}T'_{rs,r} \end{aligned}$$

But:

$$T'_{rs,r} = \frac{\partial T'_{rs}}{\partial x'^r} = 0$$

By our previous (3.2). Thus:

$$T^i_{j,i} = 0 \tag{3.7}$$

Gravitational fields are produced by a system of matter and energy. But gravitational fields produce a distortion in space and time.
Einstein therefore looked for a tensor proportional to T^i_j, which satisfies the above equation. But, we know of such a tensor: (2.31):

$$(R^i_j - \tfrac{1}{2}\delta^i_j R)_{,i} = 0$$

So, Einstein proposed his equation of gravity:

$$R^i_j - \tfrac{1}{2}\delta^i_j R = -\kappa T^i_j \tag{3.8}$$

Where κ is some constant. Also, since $\delta^i_{j,k} = 0$, Einstein added on another term:

$$R^i_j - \tfrac{1}{2}\delta^i_j R + \Lambda\delta^i_j = -\kappa T^i_j \tag{3.9}$$

Where Λ is called the cosmic constant.

3.1 Vacuum Equations

In a vacuum, $T^i_j = 0$, so that (3.8) becomes:

$$R^i_j - \tfrac{1}{2}\delta^i_j R = 0 \tag{3.10}$$

Contracting w.r.t i:

$$R^i_i - \tfrac{1}{2}\delta^i_i R = 0$$

But, $R^i_i = R$ and $\delta^i_i = 4$. Thus:

$$R - 2R = 0$$

That is, $R = 0$. Hence, (3.10) reads:

$$R^i_j = 0$$

Lowering the i:

$$g_{ik}R^i_j = R_{kj} = 0 = R_{jk}$$

Thus, in a vacuum, the equations of gravity become the vanishing of the Ricci tensor:

$$R_{jk} = 0 \tag{3.11}$$

We shall derive the tensor T^{ij}, by using (1.22):

$$\begin{aligned} T^{ij} &= \frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s}T'_{rs} \\ &= \frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s}\left\{\left(\rho'_0 + \frac{p'}{c^2}\right)u'_r u'_s + \delta_{rs}p')\right\} \\ &= \left(\rho'_0 + \frac{p'}{c^2}\right)\left(\frac{\partial x^i}{\partial x'^r}u'_r\right)\left(\frac{\partial x^j}{\partial x'^s}u'_s\right) + \frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^r}p' \end{aligned}$$

But:

$$x^i = x^i(x'_1, x'_2, x'_3, x'_4)$$

So that:

$$dx^i = \frac{\partial x^i}{\partial x'^r}dx'_r$$

Hence:

$$\frac{dx^i}{d\tau} = \frac{\partial x^i}{\partial x'^r}\frac{dx'_r}{d\tau'}$$

Where we have used that $d\tau = d\tau'$. Hence, we see that we have the 4-velocity:

$$u^i = \frac{dx^i}{d\tau}$$

Which implies:

$$u^i = \frac{\partial x^i}{\partial x'^r}u'_r$$

We also have found previously that:

$$g^{ij} = \frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^r}$$

Hence:

$$T^{ij} = \left(\rho'_0 + \frac{p'}{c^2}\right)u^i u^j + g^{ij}p'$$

Putting $\rho'_0 = \rho'_0(x') = \rho'_0(x) = \rho$ and $p' = p'(x') = p'(x) = p$, then:

$$T^{ij} = \left(\rho_0 + \frac{p}{c^2}\right)u^i u^j + g^{ij}p \tag{3.12}$$

Now:

$$T^i_j = g_{jk}T^{ik}$$

Thus:

$$T^i_j = \left(\rho_0 + \frac{p}{c^2}\right) u^i u_j + \delta^i_j p \tag{3.13}$$

Similarly:

$$T_{ij} = \left(\rho_0 + \frac{p}{c^2}\right) u_i u_j + g_{ij} p \tag{3.14}$$

Noting that $u_i = g_{ik}u^k$.

Equations (3.8), (3.9) and (3.11) are complicated non-linear differential equations, even so there are very many exact solutions (a few of which are given by D.Lawden in *An Introduction to Tensor Calculus, Relativity & Cosmology* Chapter 6). A particular solution of (3.11) is the Schwartzchild solution (see Lawden, Chapter 6). The Schwartzchild solution represents the gravitational field of a singularity of mass m. This was generalised by R.Kerr in 1963, and represents a rotating singularity of mass m and angular momentum a. When $a = 0$, the Kerr solution reduces to the Schwartzchild solution. These solutions are important because they are thought to represent the steady state solutions of a black hole.