Hamilton's Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Putting:

$$\frac{\partial L}{\partial \dot{q}_i} = p_i$$

$\therefore$ $$\frac{dp_i}{dt} - \frac{\partial L}{\partial q_i} = 0$$

$\therefore$ $$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

Now:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q}, t) = f_i(q, \dot{q}, t)$$

There are n p_i's and n $\dot{q}_i$'s.

$\therefore$ solving for $\dot{q}_i$ in terms of the other variables:

$\therefore$ $$\dot{q}_i = g_i(q, p, t) \qquad (1)$$

Let:

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \qquad (2)$$

From (1) and (2):

$$H = H(p, q, t)$$

When expressed in this form, H is called the Hamiltonian.

Using the summation convention:

$$H = H(p, q, t) = p_i \dot{q}_i - L \qquad (3)$$

Differentials:

Taylor series:

$$f(a_1 + h_1, a_2 + h_2, ..., a_n + h_n)$$

$$= f(a_1, a_2, ..., a_n) + h_1 \frac{\partial f}{\partial a_1} + h_2 \frac{\partial f}{\partial a_2} + ... + h_n \frac{\partial f}{\partial a_n} + \frac{1}{2!}\left(\begin{array}{l} \frac{\partial^2 f}{\partial a_1^2} h_1^2 + \frac{\partial^2 f}{\partial a_2^2} h_2^2 + ... + \frac{\partial^2 f}{\partial a_n^2} h_n^2 + \\ \frac{\partial^2 f}{\partial a_1 \partial a_2} h_1 h_2 + \frac{\partial^2 f}{\partial a_1 \partial a_3} h_1 h_3 + ... \end{array}\right)$$

That is:

$$f(a_1 + h_1, a_2 + h_2, ..., a_n + h_n)$$

$$= f(a_1, a_2, ..., a_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial a_i} h_i + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial a_i \partial a_j} h_i h_j$$

$$+\frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^3 f}{\partial a_i \partial a_j \partial a_k}h_i h_j h_k$$
$$+\ldots$$

$$\therefore \qquad f(a_1+h_1, a_2+h_2, \ldots, a_n+h_n) - f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n}\frac{\partial f}{\partial a_i}h_i + \ldots$$

Let:

$$f(a_1+h_1, a_2+h_2, \ldots, a_n+h_n) - f(a_1, a_2, \ldots, a_n) = df$$
$$h_i = da_i \qquad\qquad i = 1,2,\ldots,n$$

Then:

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial a_i}da_i + \ldots$$

So, to the 1st order of "small quantities":

$$df \approx \sum_{i=1}^{n} da_i \frac{\partial f}{\partial a_i}$$

We write:

$$df = \sum_{i=1}^{n} da_i \frac{\partial f}{\partial a_i}$$

df, da_i are called differentials.

$f(a_1, a_2, \ldots, a_n)$ could also be written as $f(x_1, x_2, \ldots, x_n)$, so:

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i = \frac{\partial f}{\partial x_i}dx_i$$

Now, going back to equation (3), and using the differential formulation:

$$dH = d(p_i, \dot{q}_i) - dL$$
$$= \dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt\right)$$
$$= \dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial L}{\partial q_i}dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t}dt\right)$$
$$= \dot{q}_i dp_i - \left(\dot{p}_i dq_i + \frac{\partial L}{\partial t}dt\right) \qquad (4)$$

As:

$$H(p, q, t)$$
$$\therefore \qquad dH = \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial t}dt$$
$$\therefore \qquad \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial t}dt = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t}dt$$

Now, the coefficients are equal, so:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad (5)$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad (6)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (7)$$

From (4):

$$\frac{dH}{dt} = \dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i - \frac{\partial L}{\partial t}$$

$$\therefore \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t} \qquad (8)$$

Suppose that L does not contain the time explicitly, i.e.: $L(q,\dot{q})$. Then:

$$\frac{\partial L}{\partial t} = 0$$

$\therefore$ from (8):

$$\frac{dH}{dt} = 0 \qquad (9)$$

$\therefore$ H is a constant.

But:

$$\begin{aligned} H &= p_i \dot{q}_i - L \\ &= \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\ &= \dot{q}_i \frac{\partial}{\partial \dot{q}_i}(T - V) - (T - V) \\ &= \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - (T - V) \\ &= 2T - (T - V) \\ &= T + V \end{aligned}$$

$\therefore$

$$T + V = const$$

Small Oscillations:

From before:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

We are going to take $\dot{q}_i$ to be small.

$$V = V(q) = V(0,0,...,0) + \frac{\partial V}{\partial q_i} q_i \bigg|_{q_i=0} + \frac{1}{2!} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j \bigg|_{q_i=0} + ...$$

If the body is in equilibrium at $q_i = 0$, then:

$$\frac{\partial V}{\partial q_i}\bigg|_0 = 0$$

Putting:

$$V(0,0,...,0) = c$$

So that:

$$V - c = \frac{1}{2!}\frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j \bigg|_{q=0} + \ldots$$

Replacing $V - c$ by V, then:

$$V = \frac{1}{2} b_{ij} q_i q_j$$

Where:

$$b_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_0 = const$$

$$a_{ij} = a_{ij}(q) = a_{ij}(0,0,\ldots,0) + \frac{\partial a_{ij}}{\partial q_k} q_k \bigg|_0 + \ldots$$

$\therefore$ to the second order in q_i's we have:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

$\therefore$ we can write, to the second order of small quantities:

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$$

$$V = \frac{1}{2} b_{ij} q_i q_j$$

Where a_{ij}, b_{ij} are constants.

Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

$$T = \frac{1}{2} a_{kl} \dot{q}_k \dot{q}_l$$

$$\therefore \quad \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} a_{kl} \left(\frac{\partial \dot{q}_k}{\partial \dot{q}_i} \dot{q}_l + \frac{\partial \dot{q}_l}{\partial \dot{q}_i} \dot{q}_k \right)$$

$$= \frac{1}{2}\left(d_{ki} \dot{q}_l + d_{li} \dot{q}_k\right)$$

$$= \frac{1}{2}\left(a_{il} \dot{q}_l + a_{ik} \dot{q}_k\right)$$

$$= \frac{1}{2}\left(2 a_{ik} \dot{q}_k\right)$$

$$= a_{ik} \dot{q}_k$$

$$\frac{\partial T}{\partial q_i} = 0$$

$$\frac{\partial V}{\partial q_i} = \frac{1}{2}\left(2 b_{ij} a_j\right) = b_{ij} a_j$$

$\therefore$ Lagrange gives:

$$a_{ik} \ddot{q}_k = -b_{ij} q_j$$

Or:

$$a_{ik}\ddot{q}_k + b_{ik}q_k = 0$$

Let:

$$q_k = A_k \cos(wt + e)$$

∴

$$\ddot{q}_k = -w^2 A_k \cos(wt + e)$$

So:

$$\left(-a_{ik}w^2 A_k + b_{ik}A_k\right)\cos(wt + e) = 0$$

∴

$$-a_{ik}w^2 A_k + b_{ik}A_k = 0$$

$$\left(a_{ij}w^2 - b_{ij}\right)X = 0 \qquad X = \begin{pmatrix} A_1 \\ A_2 \\ \ldots \\ A_n \end{pmatrix}$$

∴ X is an eigenvector and w^2 is an eigenvalue of the matrix:

$$a_{ij}w^2 - b_{ij}$$

∴

$$\left|a_{ij}w^2 - b_{ij}\right| = 0$$