The work done when q_i changes to $q_i + dq_i$ for $i = 1,2,...,n$ is given by:

$$W = \sum_{i=1}^{n} Q_i dq_i \qquad (1)$$

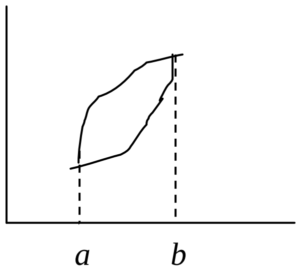
We can find Q_i from (1) when the system is not conservative.

Calculus of Variations:

Consider the integral:

$$I = \int_a^b F(x, y, y')dx \qquad y' = \frac{dy}{dx}$$

We have to find that value of $y = f(x)$ which makes I a max or min – stationary.



x and Y axes.
Bottom curve is (1), top (2)
Both go through a and b.
(1) has equation: $Y = y = f(x)$
(2) has equation $Y = y + a\,h(x)$

So, $h(a) = h(b) = 0$, hence endpoints correspond.
So that for (2):

$$I(a) = \int_a^b F(x, y + a\,h, y' + a\,h')dx$$

For a stationary value:

$$\left.\frac{dI}{da}\right|_{a=0} = 0$$

But,

$$\frac{dI}{da} = \int_a^b \left[\frac{\partial F}{\partial Y}\right] h + \left[\frac{\partial F}{\partial Y'}\right] h' dx$$

Where:

$$\left[\frac{\partial F}{\partial Y}\right] = \left.\frac{\partial F(x, Y, Y')}{\partial Y}\right|_{\substack{Y = y + a h \\ Y' = y' + a h'}}$$

$$\left[\frac{\partial F}{\partial Y'}\right] = \left.\frac{\partial F(x, Y, Y')}{\partial Y'}\right|_{\substack{Y = y + a h \\ Y' = y' + a h'}}$$

$$\therefore \qquad \left.\frac{dI}{da}\right|_{a=0} = \int_a^b \left[\frac{\partial F}{\partial Y}\right]_{a=0} h + \left[\frac{\partial F}{\partial Y'}\right]_{a=0} h' dx$$

And:

$$\left[\frac{\partial F}{\partial Y}\right]_{a=0} = \left.\frac{\partial F(x, Y, Y')}{\partial Y}\right|_{\substack{Y = y \\ Y' = y'}} = \frac{\partial}{\partial Y} F(x, y, y') = \frac{\partial F}{\partial y}$$

Similarly:

$$\left[\frac{\partial F}{\partial Y'}\right]_{a=0} = \frac{\partial F(x,Y,Y')}{\partial Y'}\bigg|_{\substack{Y=y \\ Y'=y'}} = \frac{\partial}{\partial Y'}F(x,y,y') = \frac{\partial F}{\partial y'}$$

$$\therefore \qquad \frac{dI}{da}\bigg|_{a=0} = \int_a^b \left(\frac{\partial F}{\partial y}h + \frac{\partial F}{\partial y'}h'\right)dx$$

$$= \int_a^b \frac{\partial F}{\partial y}h\,dx + \int_a^b \frac{\partial F}{\partial y'}h'\,dx$$

$$= \int_a^b \frac{\partial F}{\partial y}h\,dx + \left[\frac{\partial F}{\partial y'}h(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)h\,dx \qquad \text{(int. by parts)}$$

$$= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)h\,dx + \frac{\partial F}{\partial y'}\bigg|_{x=b} h(b) - \frac{\partial F}{\partial y'}\bigg|_{x=a} h(a)$$

But:

$$h(a) = h(b) = 0$$

$$\therefore \qquad \frac{dI}{da}\bigg|_{a=0} = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)h\,dx$$

But:

$$\frac{dI}{da}\bigg|_{a=0} = 0$$

$$\therefore \qquad \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)h\,dx = 0$$

Let $f = \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)$

Then:

$$\int_a^b fh\,dx = 0 \qquad (2)$$

If $f > 0$ for some value of x in the interval $a \le x \le b$, then by continuity there is an interval $x - d \to x + d$ for which $f > 0$.

$\therefore$ Define $\quad h = 0$ for the intervals $(a, x-d)$ and $(x+d, b)$
$\quad h > 0$ in the interval $(x-d, x+d)$

$$\therefore \qquad \int_a^b fh\,dx > 0$$

Which contradicts (2). $\therefore\ f \le 0$.

Similarly, if $f < 0$ we get a contradiction as above. $\therefore\ f \ge 0$.

$$\therefore \qquad 0 \le f \le 0$$

$$\therefore \qquad f = 0$$

$$\therefore \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Or: $$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0 \qquad (3)$$

Which is known as Euler's equation.

Generalisation:

$$I=\int_a^b F(x,y_1,y_2,...,y_n,y_1',y_2',...,y_n')dx$$

In order to make I stationary we can vary y_i, $1\le i\le n$, and just as above, we will get Euler's equation:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)-\frac{\partial F}{\partial y_i}=0 \qquad \text{for} \qquad i=1,2,...,n$$

Putting $x=t$ (time) $y_i=q_i$, $y_i'=\dot{q}_i$ for $1\le i\le n$, we get:

$$I=\int_{t_1}^{t_2} L(t,q_1,q_2,...,q_n,\dot{q}_1,\dot{q}_2,...,\dot{q}_n)dt$$

$$\therefore \qquad \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0 \qquad i=1,2,...,n$$

Hence we have Hamilton's Principle:

For any dynamic path from $t=t_1$ to $t=t_2$, the integral:

$$\int_{t_1}^{t_2} Ldt$$

Is stationary.

Consider again a dynamic system of n degrees of freedom. But each point does not depend upon time t explicitly.

i.e.

$$x=x(q_1,q_2,...q_n)$$
$$y=y(q_1,q_2,...,q_n)$$
$$z=z(q_1,q_2,...,q_n)$$

$$\therefore \qquad \underline{r}=x\underline{i}+y\underline{j}+z\underline{k}=\underline{r}(q_1,q_2,...,q_n)$$

$$\therefore \qquad \dot{\underline{r}}=\frac{\partial \underline{r}}{\partial q_i}\frac{\partial q_i}{\partial t}=\frac{\partial \underline{r}}{\partial q_i}\dot{q}_i$$

$$\therefore \qquad \dot{\underline{r}}\cdot\dot{\underline{r}}=\frac{\partial \underline{r}}{\partial q_i}\dot{q}_i\cdot\frac{\partial \underline{r}}{\partial q_j}\dot{q}_j=\left(\frac{\partial \underline{r}}{\partial q_i}\cdot\frac{\partial \underline{r}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$\therefore \qquad T=\sum_{all\_particles}\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}}$$

$$=\dot{q}_i\dot{q}_j\sum\left(\frac{\partial \underline{r}}{\partial q_i}\cdot\frac{\partial \underline{r}}{\partial q_j}\right)\frac{1}{2}m$$

Putting:

$$a_{ij} = \sum \left(\frac{\partial \underline{r}}{\partial q_i} \cdot \frac{\partial \underline{r}}{\partial q_j} \right) m$$

$\therefore$ $$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j \qquad a_{ij}(q)$$

Note that:

$$a_{ji} = \sum \left(\frac{\partial \underline{r}}{\partial q_j} \cdot \frac{\partial \underline{r}}{\partial q_i} \right) m = a_{ij}$$

$\therefore$
$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_k} &= \frac{1}{2} a_{ij} \frac{\partial}{\partial \dot{q}_k} \left(\dot{q}_i \dot{q}_j \right) \\
&= \frac{1}{2} a_{ij} \left(\dot{q}_i \frac{\partial \dot{q}_j}{\partial \dot{q}_k} + \dot{q}_j \frac{\partial \dot{q}_i}{\partial \dot{q}_k} \right) \\
&= \frac{1}{2} a_{ij} (\dot{q}_i d_{jk} + \dot{q}_j d_{ik}) \\
&= \frac{1}{2} (a_{ij} \dot{q}_i d_{jk} + a_{ij} \dot{q}_j d_{ik}) \\
&= \frac{1}{2} (a_{ik} \dot{q}_i + a_{jk} \dot{q}_j) \\
&= \frac{1}{2} (a_{ik} \dot{q}_i + a_{ik} \dot{q}_i) \\
&= a_{ik} \dot{q}_i
\end{aligned}$$

$\therefore$ $$\dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = a_{ik} \dot{q}_i \dot{q}_k = a_{ij} \dot{q}_j \dot{q}_i = 2T$$

$\therefore$

$$\dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T$$

Theorem:

If T and V do not contain the time explicitly, so that L does not contain the time explicitly, the Lagrange equations has an integral:

$$T + V = \text{constant}$$

Proof:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

$$L = L(q, \ldots, \dot{q}, \ldots) = L(q, \dot{q}) \qquad \text{(notation)}$$

$\therefore$ $$\frac{dL}{dt} = \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \qquad (4)$$

But:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad\qquad f_i = \frac{\partial L}{\partial \dot{q}_i} = f(q,\dot{q})$$

$$= \frac{d}{dt} f_i(q,\dot{q})$$

$$= \frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial \dot{q}_j}\ddot{q}_j$$

$$\therefore \quad \frac{\partial L}{\partial q_i} = \dot{q}_j \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} + \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \qquad (5)$$

$$\frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \dot{q}_i \frac{\partial}{\partial q_i}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$= \dot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} + \ddot{q}_i \left(\frac{\partial \dot{q}_j}{\partial \dot{q}_i}\frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$$

$$= \dot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} + \ddot{q}_i \left(d_{ij}\frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$$

$$= \dot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} + \ddot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i} + \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$$

$$= \dot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

$$= \dot{q}_i \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial q_j} + \ddot{q}_j \dot{q}_i \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$= \dot{q}_i \left(\dot{q}_j \frac{\partial^2 L}{\partial q_i \partial q_j} + \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\right) + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

The part in brackets is (5):

$$\therefore \quad \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

Comparing with (4) $= \dfrac{dL}{dt}$

$$\therefore \quad \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) = 0$$

$$\therefore \quad \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = const$$

But, $L = T - V$ where V is the potential function$= V(q_1,...,q_n)$

$$\therefore \quad \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$\therefore \qquad \dot{q}_j \dfrac{\partial T}{\partial \dot{q}_j} - (T - V) = const$

But:

$$\dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T$$

From above.

$\therefore \qquad 2T - (T - V) = const$

$\therefore \qquad T + V = const$

The constant is the total energy E:

$$T + V = E$$