Electromagnetism

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1 Electrostatics

1.1 Coulomb's Law

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r} \tag{1}$$

1.2 Electric Potential

From $W = q\int_a^b \mathbf{E}\cdot d\mathbf{l}$, we write the potential difference:

$$\phi(r_a) - \phi(r_b) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} \tag{2}$$

And the potential in going from infinity to r:

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r} \tag{3}$$

We have:

$$\mathbf{E} = -\nabla\phi \tag{4}$$

In electrostatic fields only, as $\mathbf{E}$ is conservative, we have $\nabla\times\ bfE = 0$.

1.3 Electric Dipole

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right) \tag{5}$$

For the off-axis potential, for two charges $\pm q$, seperated by d.

1.4 Gauss' Law

1.4.1 Integral Form

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\int_V \rho dV \tag{6}$$

Where the closed surface S encloses the volume V.

1.4.2 Differential Form

From the integral form, we use the divergence theorem:

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{a} \, dV \tag{7}$$

$$\Rightarrow \int_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} \, dV = \frac{1}{\epsilon_0} \int_V \rho \, dV \tag{8}$$

And as we can shrink the volume down to a point, we result in:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{9}$$

This is Maxwell's first equation.
As $\mathbf{E} = -\nabla\phi$, we can also write Poisson's equation, and Laplace's equation (i.e. if no charges present):

$$\nabla^2\phi = \frac{\rho}{\epsilon_0} \tag{10}$$

$$\nabla^2\phi = 0 \tag{11}$$

1.5 Capacitors & Dielectrics

$$C = \frac{Q}{V} \tag{12}$$

Find the potential difference V from $\int_a^b \mathbf{E} \cdot d\mathbf{l}$.

A dielectric material is an insulator: thus conductivity is zero.

Dielectrics gain dipole moments when places in an electric field. i.e. they polarise. $C = \epsilon C_0$. As dielectric inserted, capacitance increases, thus reducing the internal electric field.

1.6 Polarisation

1.6.1 Mechanisms for Polarisation:

Electronic: The atoms within the dielectric shift, their electron clouds moving, hence leaving the atom with a dipole, which then aligns with the applied electric field.
Aligned Dipoles: Works in a similar way, but this time the intrinsic dipoles of the molecules align - when there is no external field present, they are at 'random'.

1.6.2 Dipole Moment

The dipole moment $\mathbf{p}$ due to two charges q seperated by $\mathbf{d}$ is given by:

$$\mathbf{p} = q\mathbf{d} \tag{13}$$

The total polarisation **P** is therefore given by:

$$\mathbf{P} = n\mathbf{p} \tag{14}$$

basically the total number of dipole moments.
The surface charge density is given by the dot product of the polarisation with the outward normal of the surface:

$$\sigma_p = \mathbf{P} \cdot \hat{\mathbf{n}} \tag{15}$$

We have the following relations between polarisation **P**, electric field **E**, electric susceptibility χ_E & polarisation charge density ρ_p:

$$\mathbf{P} = (\epsilon - 1)\epsilon_0 \mathbf{E} \tag{16}$$
$$\chi_E = \epsilon - 1 \tag{17}$$
$$\rho_p = -\nabla \cdot \mathbf{P} \tag{18}$$

In writing (16), we assumed: linearity, isotropy, homengeity & non-conducting.

1.7 Electric Displacement Vector

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{19}$$

Thus, we have a new version of Gauss' law:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{20}$$
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f \, dV \tag{21}$$

Where ρ_f denotes free charges.
D is not affected by polarisation charges, and **P** exists only within the dielectric.
The total charge density is the sum of the free charge density with the polarisation charge density:

$$\rho = \rho_p + \rho_f \tag{22}$$

1.7.1 Boundary Conditions

D_{perp} is continuous across boundaries.
$E_{//}$ is continuous across boundaries.

That is:

$$D_1 \cos\theta_1 = D_2 \cos\theta_2 \tag{23}$$
$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \tag{24}$$

1.7.2 Energy Density

Can be given by:

$$U = \int_V \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, d\tau \tag{25}$$

2 Magnetostatics

2.1 Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{26}$$

So, if $\mathbf{E} = 0 \Rightarrow \mathbf{F} = q\mathbf{v} \times \mathbf{B}$.
So, if a charge is moving solely in the presence of a magnetic field, we have $\mathbf{F}$ being perpendicular to velocity, hence motion in a circle, with period τ:

$$F = \frac{mv^2}{r} = qvB \tag{27}$$

$$\tau = \frac{2\pi r}{v} = \frac{2\pi m}{qB} \tag{28}$$

Hence, we define the cyclotron frequency $\nu = \frac{1}{\tau}$ as:

$$\nu = \frac{qB}{2\pi m} \tag{29}$$

2.2 Maxwell's 2nd Equation

$$\nabla \cdot \mathbf{B} = 0 \tag{30}$$

2.3 Current Density

Current density is the number of charges, times the charge, times the drift velocity of the charges. Alternatively, it is the current per unit area:

$$j = \frac{I}{A} = -Nev_d \tag{31}$$

$$\Rightarrow I = \int_S \mathbf{j} \cdot d\mathbf{A} \tag{32}$$

Ohm's law becomes:

$$\mathbf{j} = \sigma\mathbf{E} \tag{33}$$

Where σ is electrical conductivity.

2.4 The Hall Effect

Current **I** flows along a slab of conductor, with width w and depth d - hence area A. A magnetic field **B** intersects the slab, so that **B** is perpendicular to **I**. Hence, the charge carriers 'feel' a force. The charge carriers are electrons, with drift velocity v_d.
We define:
$F_m \equiv$ magnetic force on negative charge carriers;
$F_e \equiv$ electric force on charge build up.
So, we begin by writing the force, and current:

$$\begin{aligned} F_e &= eE && (34) \\ F_m &= ev_d B && (35) \\ I &= nev_d A && (36) \\ \Rightarrow v_d &= \frac{I}{neA} && (37) \\ \Rightarrow F_m &= \frac{eIB}{neA} && (38) \end{aligned}$$

And, to find the potential in equilibrium (i.e. $F_e = F_m$):

$$F_e = eE = \frac{V_H e}{w} \tag{39}$$

Where the potential is $V_H = wE$. Thus, the system is in equilibrium when:

$$\frac{V_H e}{w} = \frac{eIB}{neA} \tag{40}$$

Therefore, the potential set up, the 'Hall Voltage' in equilibrium, across the slab is:

$$V_H = \frac{IB}{ned} \tag{41}$$

2.5 Force on a Wire

Force on one electron, by Lorentz:

$$\mathbf{f} = -e\mathbf{v} \times \mathbf{B} \tag{42}$$

If the conductor has length $d\mathbf{l}$, and area $\mathbf{A}$; thus, $nAdl$ electrons, at a velocity v_d. Hence, the total force:

$$d\mathbf{F} = -(nAdl)e\mathbf{v_d} \times \mathbf{B} \tag{43}$$

But, $I = -neAv_d$. Thus:

$$d\mathbf{F} = Id\mathbf{l} \times \mathbf{B} \tag{44}$$

2.6 Ampere's Law

2.6.1 Integral Form

$$\oint_\ell \mathbf{B} \cdot d\mathbf{l} = \mu_0 \sum I \tag{45}$$

2.6.2 Differential Form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{46}$$

If the integral version is used on a long straight wire, of radius r:

$$\oint_\ell \mathbf{B} \cdot d\mathbf{l} = 2\pi r B = \mu_0 I \tag{47}$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r} \tag{48}$$

2.7 Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{49}$$

2.8 Magnetostatics in Materials

Inductance in a vacuum is $L_0 = \mu_0 N^2 \pi r^2 l$, and in materials is $L = \mu L_0$; where $\mu = \frac{1}{1-\chi_B}$.

2.8.1 Magnetic Dipoles

Magnetic moment = current × area:

$$\mathbf{m} = I\mathbf{A} \tag{50}$$

Torque on a current loop:

$$\tau = \mathbf{m} \times \mathbf{B} \tag{51}$$

Potential energy of current loop:

$$U(\theta) = -\mathbf{m} \cdot \mathbf{B} \tag{52}$$

Magnetisation is like polarisation: total dipole moment per unit area:

$$\mathbf{M} = N\mathbf{m} \tag{53}$$

Thus, we have:

$$\mathbf{M} = \chi_B \frac{\mathbf{B}}{\mu_0} \tag{54}$$

2.8.2 Forms of Magnetisation

Diamagentism: From individual atoms, and electrons orbiting in **B** fields. Weak. Linear. Negative.
Paramagnetism: From moments aligning. Stronger. Linear. Positive.
Ferromagnetism: Due to domains within material. Strong. Non-linear. Permenant. Has hysterisis.

Magnetic dipoles in materials can be visualised as small current loops, where the internal components cancel out, to leave only a surface current.

As before, we have:

$$\begin{aligned} \mathbf{i}_s &= \mathbf{M} \times \hat{\mathbf{n}} && (55) \\ \mathbf{i}_b &= \nabla \times \mathbf{M} && (56) \\ \Rightarrow \mathbf{i} &= \mathbf{i}_s + \mathbf{i}_b && (57) \\ \mathbf{B} &= \mathbf{B}_0 + \mu_0 \mathbf{M} && (58) \end{aligned}$$

2.9 Magnetic Field Vector

We have:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (59)$$

Thus, we have a new Ampere's law, involving free-currents:

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{j}_f && (60) \\ \oint \mathbf{H} \cdot d\mathbf{l} &= I_f && (61) \end{aligned}$$

Energy in magnetic fields is thus:

$$U = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \quad (62)$$

H-field lines can be discontinuous:

$$\nabla \cdot \mathbf{H} = -\nabla \times \mathbf{M} \quad (63)$$

And, at boundaries we have that B_{perp} and $H_{//}$ are conserved.

3 Maxwell's Equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} && (64) \\ \nabla \cdot \mathbf{B} &= 0 && (65) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && (66) \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (67) \end{aligned}$$

With corresponding alternatives:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{68}$$
$$\nabla \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}$$

4 EM Waves in Vacuum

Maxwell's equations in free space read:

$$\nabla \cdot \mathbf{E} = 0 \tag{70}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{71}$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{72}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{73}$$

We can easily derive the wave equation:

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{74}$$

Where the speed is given by $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$. (74)

Now, solutions to wave equations have the form $f(z - vt)$; thus we can write things like plane wave solutions $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$.

We write the relation between **E** and **B**:

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \tag{76}$$
$$\Rightarrow cB_y = E_x \tag{77}$$

Energy density in the wave is given by:

$$U = \frac{1}{2} \int_V (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \, d\tau \tag{78}$$

Or, equivalently:

$$U = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \tag{79}$$
$$= \epsilon_0 E_0^2 \cos^2(kz - \omega t) \tag{80}$$

Hence, electric and magnetic fields carry the same amount of energy.

4.1 Poynting Vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \mathbf{E} \times \mathbf{H} \tag{81}$$
$$\Rightarrow S = \frac{1}{\mu_0 c} E_0^2 \tag{82}$$

4.1.1 Impedance of Free Space

$$= \left(\sqrt{\frac{\epsilon_0}{\mu_0}}\right)^{-1} = 377\Omega \tag{83}$$

4.1.2 Irradiance

A time averages Poynting vector:

$$I =< \mathbf{S} >= \frac{E_0^2}{2\mu_0 c} \tag{84}$$

4.1.3 Radiation Pressure

If radiation absorbed or reflected:

$$P^{abs} = \frac{I}{c} \tag{85}$$

$$P^{ref} = \frac{2I}{c} \tag{86}$$

So, the force exterted due to radiation pressure is $F = PA$, the standard pressure times area.

5 Polarisation States

5.1 Plane/Linear Polarisation

Electric field is confined to a plane:

$$\mathbf{E} = E_0 \cos(kz - \omega t)\mathbf{i} \tag{87}$$

5.2 Circular/Elliptical Polatisation

$$\mathbf{E} = E_x \cos(kz - \omega t)\mathbf{i} + E_y \cos(kz - \omega t + \delta)\mathbf{j} \tag{88}$$

If $E_x = E_y$ & $\delta = \pm\frac{\pi}{2}$, then the wave is circularly polarised. Anything else gives elliptical polarisation.

6 EM Waves in Materials

Here we have $\epsilon_0 \rightarrow \epsilon_0\epsilon$ and $\mu_0 \rightarrow \mu\mu_0$. We still assume no charges/currents. The wave equation becomes:

$$\nabla^2\mathbf{E} = \mu\mu_0\epsilon\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{89}$$

Now, the velocity of the wave is:

$$
\begin{aligned}
v &= \frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}} = \frac{c}{\sqrt{\mu\epsilon}} = \frac{c}{n} &(90)\\
\Rightarrow n &= \sqrt{\mu\epsilon} \approx \sqrt{\epsilon} &(91)
\end{aligned}
$$

We have Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, but generally is:

$$
k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \tag{92}
$$

For an incident wave $\mathbf{E}_I$, with reflection $\mathbf{E}_R$, and transmission $\mathbf{E}_T$. (92)

Now, boundary conditions state:

$$
\begin{aligned}
E_I + E_R &= E_T &(94)\\
H_I + H_R &= H_T &(95)\\
\Rightarrow \frac{B_I}{\mu_0} - \frac{B_R}{\mu_0} &= \frac{B_T}{\mu\mu_0} &(96)
\end{aligned}
$$

But, we also have that $B = \frac{1}{c}E = \sqrt{\epsilon_0\mu_0}E$, thus:

$$
E_I\sqrt{\frac{\epsilon_0}{\mu_0}} - E_R\sqrt{\frac{\epsilon_0}{\mu_0}} = E_T\sqrt{\frac{\epsilon\epsilon_0}{\mu\mu_0}} \tag{97}
$$

Which, using the approximation that $\mu \approx 1$:

$$
E_I - E_R = nE_T \tag{98}
$$

Thus, we define the reflection and transmission coefficients thus:

$$
\begin{aligned}
R &= \frac{E_R^2}{E_I^2} = \left(\frac{1-n}{n+1}\right)^2 &(99)\\
T &= \frac{E_T^2}{E_I^2}\frac{v}{c} = \frac{4n}{(1+n)^2} &(100)
\end{aligned}
$$

Noticing that $T + R = 1$

7 EM Waves in a Conducting Medium

Here we have $\mathbf{j} = \sigma\mathbf{E}$. Thus, with this expresion for current density, Maxwell 3 becomes:

$$
\nabla \times \mathbf{B} = \mu_0\sigma\mathbf{E} + \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} \tag{101}
$$

However, in a 'good conductor', we have that $\mu_0\sigma\mathbf{E} >> \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$. Hence, we have:

$$
\nabla \times \mathbf{B} = \mu_0\sigma\mathbf{E} \tag{102}
$$

We can also derive the skin depth attenuation factor: $\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}}$.

7.1 Plasmas

Using: $\mathbf{F} = m\ddot{\mathbf{r}} = q\mathbf{E}$ and $\mathbf{j} = Ne\dot{\mathbf{r}}$, we can derive:

$$\nabla^2\mathbf{E} = \mu_0 n_e e \left(\frac{e\mathbf{E}}{m_e}\right) + \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{103}$$

From which we can derive that below the plasma frequency $\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$, incident EM waves are strongly attenuated.