# Complex Variables & Integral Transforms

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## Contents

# 1 Complex Variables

## 1.1 General Relations & Definitions

$$z = x + iy = re^{i\theta} \tag{1}$$
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$
$$z^n = (re^{i\theta})^n = r^n(\cos n\theta + i\sin n\theta) \tag{3}$$
$$\overline{z} = x - iy \tag{4}$$
$$z = \overline{z}z \tag{5}$$
$$e^{i\pi} + 1 = 0 \tag{6}$$

*Principle value range*: range within which functions are well defined/single valued.
*Open*: in a set of complex numbers, if one can draw an arbitrarily small circle around any point, with the circle still in the set, then the set is open.
*Connected*: if two points in a set can be connected with a continuous curve, then the set is connected. If the set is not-empty and open and connected, then the set is a *domain*.

### 1.1.1 Hyperbolics, Sinusoidals & Logarithms

$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{7}$$
$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{8}$$
$$\cosh iz = \cos z \tag{9}$$
$$\sinh iz = i\sin z \tag{10}$$
$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta \tag{11}$$

### 1.1.2 Complex Functions

$$f(z) = u(x,y) + iv(x,y) \tag{12}$$

So that $\Re\{f(z)\} = u(x,y)$ and $\Im\{f(z)\} = v(x,y)$

### 1.1.3 Continuity & Differentiability

A complex function $f(z)$ is continuous if

$$|f(z) - f(z_0)| \to 0 \tag{13}$$

as $z \to z_0$ in any manner.

A complex function $f(z)$ is differentiable if

$$\lim_{z \to z_0} \left( \frac{f(z) - f(z_0)}{z - z_0} \right) \to 0 \tag{14}$$

in any manner.

## 1.2 Cauchy-Riemann Equations

The complex function $f(z) = u(x, y) + iv(x, y)$ is regular (analytic) if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{15}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{16}$$

hold. The functions $u(x, y)$ and $v(x, y)$ are known as 'conjugate functions', and are harmonic:

$$\nabla^2 u = \nabla^2 v = 0 \tag{17}$$

## 1.3 Laurent Series

Useful to know the following 'common' expansions:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \tag{18}$$

$$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \tag{19}$$

the first of which is convergent on $|z| < 1$, and the second on $|z| > 1$.
Now, $\sum a_n z^n$ converges on $|z| < R_2$, and $\sum b_n z^{-n}$ on $|z| > R_1$. Hence:

$$\sum a_n z^n + \sum b_n z^{-n} \tag{20}$$

converges on $R_1 < |z| < R_2$.
We have:

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz \qquad n = 0, 1, 2, \dots \tag{21}$$

$$b_n = \frac{1}{2\pi i} \oint_\gamma (z - z_0)^{n-1} f(z) \, dz \qquad n = 1, 2, 3, \dots \tag{22}$$

So, generally:

$$f(z) = \sum_{-\infty}^{\infty} A_n (z - z_0)^n \tag{23}$$

is a Laurent expansion about the point $z = z_0$.

### 1.3.1 Singularities

If, in the Laurent expansion, all coefficients of the $\frac{1}{z^n}$ terms (i.e. the $b_n$) are zero, then $z = z_0$ is a removable singularity.
If, for $n > k$, $b_n = 0$, then there is a pole of order $k$ at $z = z_0$. $k = 1$ is known as a 'simple pole', with $k = 2$ being a 'double pole'.
If there is an infinite number of $b_n$'s, then an essential singularity at $z = z_0$.
These have all been examples of 'isolated singularities'

A non-isolated is one in which in any small region surrounding one singularity, there is always another. For example $f(z) = \frac{1}{\sin \frac{1}{z}}$ has an 'essential singularity' at $z = \frac{1}{n\pi}$.
A 'branch point' singularity has an example of the logarithm: it is only uniquely defined in the cut plane, and is singular everywhere on the branch point.

$f(z)$ has a simple pole at $z = z_0$ if $f(z_0) = 0$ and $f'(z_0) \neq 0$.
$f(z)$ has a double pole at $z = z_0$ if $f(z_0) = f'(z_0) = 0$ and $f''(z_0) \neq 0$.
$g(z)$ has a pole of order $k$ at $z = z_0$, and $f(z)$ has a zero of order $k$. Where $g(z) = \frac{1}{f(z)}$.

## 1.4 Complex Integration

### 1.4.1 Integration by Parameterisation

Here, we parameterise $z$ in terms of some function of $t$:

$$\int_\gamma f(z)\, dz = \int_{t=\alpha}^{t=\beta} f\{z(t)\} \frac{dz}{dt}\, dz \tag{24}$$

### 1.4.2 Cauchy's Theorem

If $f(z)$ is regular inside some closed contour $\gamma$, then:

$$\oint_\gamma f(z)\, dz = 0 \tag{25}$$

### 1.4.3 Cauchy's Integral Formula

If $f(z)$ is regular in some domain $D$, and $\gamma$ is some closed Jordan contour in $D$; and $z_0$ inside $\gamma$, then:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz \tag{26}$$

### 1.5 Cauchy's Residue Theorem

If $f(z)$ is regular everywhere inside a closed contour $\gamma$, except at a finite number of isolated singularities $z_k$. Then:

$$\oint_{\gamma} f(z)\, dz = 2\pi i \sum_{z_k \; inside \; \gamma} Res\{f(z); z = z_k\} \tag{27}$$

Where:

$$Res\{f(z); z = z_0\} = b_1 \tag{28}$$

Thus, the residue of $f(z)$ at a singlarity at $z = z_0$, is the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent expansion.

#### 1.5.1 Finding Residues

We can find residues by a number of methods:

Let:

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \tag{29}$$

Where $f(z)$ has a pole of order $m$ at $z = z_0$. If $\phi(z_0) \neq 0$, and $\phi(z_0)$ is regular we have the following cases:

For $m = 1 \Rightarrow$ simple pole:

$$Res\{f(z); z = z_0\} = \phi(z_0) \tag{30}$$

For $m = 2 \Rightarrow$ double pole:

$$Res\{f(z); z = z_0\} = \left(\frac{\partial \phi}{\partial z}\right)_{z=z_0} \tag{31}$$

For $m = m \Rightarrow$ pole order $m$:

$$Res\{f(z); z = z_0\} = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1} \phi}{\partial z^{m-1}}\right)_{z=z_0} \tag{32}$$

Another method:
Let

$$f(z) = \frac{p(z)}{q(z)} \tag{33}$$

where both $p(z)$ & $q(z)$ are regular at $z = z_0$.
$q(z_0)$ is a zero of order $m$; and if $p(z_0) \neq 0$, then:

$$Res\{f(z); z = z_0\} = \frac{p(z_0)}{q'(z_0)} \tag{34}$$

And, if $p(z_0) = 0$, and $p'(z_0) \neq 0$ & $q''(_0) \neq 0$:

$$Res\{f(z); z = z_0\} = 2\frac{p'(z_0)}{q''(z_0)} \tag{35}$$

### 1.6 Evaluation of Real Integrals

Use to solve integrals of the form:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \tag{36}$$

Now, $z = re^{i\theta} = \cos\theta + i\sin\theta$. Thus:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{37}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \tag{38}$$

$$d\theta = \frac{dz}{iz} \tag{39}$$

Thus, the integral can be changed to a complex integral about the unit circle $\oint_C g(z)\, dz$, which can be solved.

### 1.7 Jordan's Lemma

Let $\gamma_R$ be the semi-circular contour extending from $-R \rightarrow +R$, and $\gamma_1$ the portion of the real line joining these two points. Thus, the closed semi-circular contour is $\gamma = \gamma_R + \gamma_1$.
Then, for the integral:

$$\oint_\gamma f(z)e^{imz}\, dz \tag{40}$$

That is, the integral::

$$\oint_\gamma f(z)e^{imz}\, dz = \int_{\gamma_R} f(z)e^{imz}\, dz + \int_{\gamma_1} f(z)e^{imz}\, dz \tag{41}$$

$$= \int_{\gamma_R} f(z)e^{imz}\, dz + \int_{-R}^{R} f(z)e^{imz}\, dz \tag{42}$$

If the following conditions hold: $f(z)$ only has simple poles in the finite part of the upper-half-plane; $\lim_{|z|\rightarrow\infty} f(z) = 0$; $m > 0$; Jordan's Lemma states that:

$$\lim_{R\rightarrow\infty} \int_{\gamma_R} f(z)e^{imz}\, dz = 0 \tag{43}$$

$$\tag{44}$$

So that the integral becomes:

$$\lim_{R\to\infty}\oint_\gamma f(z)e^{imz}\,dz = \lim_{R\to\infty}\int_{\gamma_1} f(z)e^{imz}\,dz \tag{45}$$

$$= \int_{-\infty}^{\infty} f(z)e^{imz}\,dz \tag{46}$$

$$= \int_{-\infty}^{\infty} f(x)e^{imx}\,dx \tag{47}$$

We use this in the evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$, by working backwards:

$$\int_{-\infty}^{\infty} f(x)\,dx = \oint_\gamma f(z)e^{imz}\,dz = \frac{1}{2\pi i}\sum_k Res\{f(z)e^{imz}; z = z_k\} \tag{48}$$

Which is therefore evalulated by Cauchy's Residue theorem. Remembering that it is the sum over residues enclosed by the countour: i.e. over all residues due to all singularities in the upper-half-plane.

# 2 Integral Transforms

## 2.1 Laplace Transform

$$\overline{f}(p) = L\{f(t)\} = \int_0^\infty f(t)e^{-pt}\,dt \tag{49}$$

The transform is a linear operator:

$$L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\} \tag{50}$$

Some common transforms:

| $f(t)$ | $\overline{f}(p) = L\{f(t)\}$ |
|---|---|
| $e^{\alpha t}$ | $\frac{1}{p-\alpha}$ |
| $t^n$ | $\frac{n!}{p^{n+1}}$ |
| $\frac{dy}{dt}$ | $p\overline{y}(p) - y(0)$ |
| $\frac{d^2y}{dt^2}$ | $p^2\overline{y}(p) - y'(0) - py(0)$ |
| $e^{i\omega t}$ | $\frac{p+i\omega}{p^2+\omega^2}$ |

## 2.2 Bromwich Contour

$$f(t) = L^{-1}\{\overline{f}(p)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f}(p)e^{pt}\,dp \tag{51}$$

This inversion integral is directly analogous to Jordan's Lemma. $c$ is chosen to the right of any singularities of $\overline{f}e^{pt}$. Only works if $\overline{f}$ has a finite number of isolated poles in the left-half-plane, $\lim_{p\to\infty}|\overline{f}| = 0$ and $m > 0$. That is:

$$f(t) = \frac{1}{2\pi i}\left(2\pi i \sum_{z_k\ in\ LHP} \overline{f}e^{pt}; z = z_k\right) \tag{52}$$

## 2.3 Fourier Transforms

### 2.3.1 Derivation of Fourier Transforms

Suppose we expand $f(x)$ on some range $\frac{\lambda}{2} \leq x \leq \frac{\lambda}{2}$ as a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + b_n \sin(k_n x) \tag{53}$$

Where:

$$k_n \equiv \frac{2\pi n}{\lambda} \tag{54}$$

$$a_0 = \frac{1}{\lambda}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)\, dt \tag{55}$$

$$a_n = \frac{2}{\lambda}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)\cos(k_n t)\, dt \tag{56}$$

$$b_n = \frac{2}{\lambda}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)\sin(k_n t)\, dt \tag{57}$$

$$\tag{58}$$

Now, using the identity $\cos[k_n(t-x)] = \cos(k_n t)\cos(k_n x) + \sin(k_n t)\sin(k_n x)$, we get:

$$f(x) = \frac{1}{\lambda}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)\, dt + \frac{2}{\lambda}\sum_{n=1}^{\infty}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)\cos[k_n(t-x)]\, dt \tag{59}$$

Notice:

$$\cos[k_n(t-x)] = \frac{1}{2}\left[e^{ik_n(t-x)} + e^{-ik_n(t-x)}\right] \tag{60}$$

$$k_n \equiv \frac{2\pi n}{\lambda} \Rightarrow k_{-n} = \frac{2\pi(-n)}{\lambda} = -k_n \tag{61}$$

$$\Rightarrow k_{-n} = -k_n \tag{62}$$

Thus, by changing the summation limits to $-\infty \leq n \leq \infty$:

$$f(x) = \frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(t)e^{ik_n(t-x)}\, dt \tag{63}$$

Now, to tidy up; we define $h \equiv \frac{2\pi}{\lambda} \Rightarrow k_n = nh$:

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} h \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) e^{-inh(t-x)}\, dt \tag{64}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} hg(nh) \tag{65}$$

$$g(nh) \equiv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) e^{-inh(t-x)}\, dt$$

(67)

Now, as $h \to 0$:

$$\lim_{h \to 0} \sum_{-\infty}^{\infty} hg(nh) = \int_{-\infty}^{\infty} g(k)\, dk \tag{68}$$

Thus:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)\, dk \tag{69}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{ik(t-x)}\, dtdk \tag{70}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk \tag{71}$$

Where:

$$F(k) \equiv \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \tag{72}$$

### 2.3.2 Summary of Fourier Transform

Transform:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \tag{73}$$

Inverse:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk \tag{74}$$