Field Theoretic Models of Dark Energy

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Abstract

We present a review of the current state of affairs of models of dark energy. Universe models containing only matter and radiation are unable to provide the observed universal acceleration – this is remedied by introducing another component to the content of the universe, dubbed dark energy. We review some of the field theories which have been proposed as models of dark energy: the cosmological constant, quintessence, K-essence and the elastic dark energy model. We discuss topological defects from a cosmological perspective, with the focus being on the properties of, and models that produce, domain walls. The aim is to give an account of the elastic dark energy model, which uses a frustrated domain wall network, utilizing the equation of state $w_{\rm dw} = v^2 - \frac{2}{3}$.

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I. THE NEED FOR DARK ENERGY

Following recent supernovae observations, it has become apparent that the universe is accelerating in its expansion [1, 2]. Observations use "standard candles", which are type 1a supernovae (SNe 1a), the result of a collapse due to electron degeneracy pressure. Such a collapse occurs at the Chandrasekhar mass, $1.4M_\odot$, calculable from well understood physics. This "known mass" allows the intrinsic brightness to be known, and the difference in observed and intrinsic allows the redshift of an observed collapse to be computed, and hence the recession speed as a function of redshift calculated. Observations of such high redshift supernovae gave the evidence that the universe is currently accelerating, but has only just started accelerating. [3]

A rather heuristic way of imagining the situation of acceleration is as follows. Given two galaxies, the space between the two galaxies is being stretched, whilst the galaxies themselves are not moving in space – indeed, the internal structure of the galaxies is not affected by the stretching of space, as the internal attractive "binding" is stronger on those "smaller" scales than the repulsive dark energy. Immediately one must pose the question: "what is doing the stretching?".

The purpose of this review is to give an understanding of the theoretical formalism employed in models of dark energy, and indeed, the formalism of general relativity into which a model is inserted.

From the point of view of general relativity, the content of a spacetime is described as components in the energy-momentum tensor $T_{\mu\nu}$, whose structure is

$$T_{\mu\nu} = \text{energy density} \oplus \text{pressure}.$$

The energy-momentum tensor contains all information regarding the distribution of "stuff" in a spacetime. For example, for our universe this includes baryonic matter, dark matter and photons; in a Schwarzschild spacetime the mass of the object (rather simplistically, this models the Sun at the centre of the Solar System). Going back to generalities, if the spacetime has multiple constituents, then one merely adds all the contributions from all the species,

$$\rho = \sum_{\text{species}} \rho_i, \qquad p = \sum_{\text{species}} p_i.$$

General relativity takes this distribution of "stuff" and equates to the geometry of the spacetime. That is, the distribution feeds the curving (and, indeed, the dynamics) of the spacetime. The gravitational action reads

$$S_{\mathrm{E}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\,(R + 2\Lambda - 2\kappa\mathcal{L}_{\mathrm{mat}}),$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $g = \det g_{\mu\nu}$, Λ some constant (the cosmological constant) and $\mathcal{L}_{\rm mat}$ the Lagrangian describing matter fields. Varying the action with respect to the metric gives the Einstein equation

$$G_{\mu\nu} = \kappa T_{\mu\nu} + \Lambda g_{\mu\nu},$$

where the Einstein and energy-momentum tensors are respectively given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \qquad T_{\mu\nu} = \mathcal{L}_{\rm mat}g_{\mu\nu} - 2\frac{\delta\mathcal{L}_{\rm mat}}{\delta g^{\mu\nu}}. \tag{1}$$

The constant of proportionality, κ is found in terms of Newton's constant G by requiring that the general theory reduces to the Newtonian theory in the non-relativistic limit (one finds $\kappa = 8\pi G$) [4]. Energy-momentum conservation, $\nabla^\mu T_{\mu\nu} = 0$, is satisfied "automatically", through the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. The Ricci tensor $R_{\mu\nu}$ and scalar R contains derivatives, in various forms, of the metric $g_{\mu\nu}$:

$$\begin{aligned}
R &= R^\mu{}_\mu, \\
R_{\mu\nu} &= R^\alpha{}_{\mu\alpha\nu}, \\
R^\rho{}_{\lambda\mu\nu} &= \partial_\mu\Gamma^\rho{}_{\lambda\nu} - \partial_\nu\Gamma^\rho{}_{\lambda\mu} + \Gamma^\rho{}_{\mu\beta}\Gamma^\beta{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\beta}\Gamma^\beta{}_{\mu\lambda}, \\
\Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\right).
\end{aligned}$$

The metric can be thought about as the part describing the geometry of a manifold. For example, taking $g_{\mu\nu} \to \delta_{ij}$, one reproduces flat Euclidean space. The next standard metric to take is the Minkowski metric, $g_{\mu\nu} \to \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Again, $\eta_{\mu\nu}$ describes flat space, with η_{00} describing the time-coordinate and η_{ij} the spatial coordinates. The metric allows one to define a notion of distance within a manifold, via

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu.$$

In writing this, one can perhaps notice that by using δ_{ij} as the metric, one reproduces Pythagoras' theorem for the distance between two points in flat Euclidean space. Thus, this line element $\mathrm{d}s^2$ for a general metric $g_{\mu\nu}$ can be thought of as a generalisation of Pythagoras' theorem.

Rather than talk about rather arbitrary manifolds, we shall specialise to the cosmological "standard model" of our universe, which uses the Friedmann-Robertson-Walker (FRW) metric, such that the line element is written

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right\}.$$

This line element is the distance in a manifold which has the properties of global isotropy, homogeneity, where the space has constant curvature k and the spatial coordinates are expanding due to the scale factor $a(t)$. The constant curvature k can be interpreted as follows, under a suitable rescaling. If $k = 0$, the spatial line element is exactly that of a flat geometry, if $k = +1$ of a closed geometry (such as the surface of a sphere) and $k = -1$ of open geometry (such as the surface of a saddle). So, this FRW metric is the geometrical model of the universe. To complete the model we must specify the components of $T_{\mu\nu}$ and solve the resulting Einstein equations. The standard assumption is that the universe is a perfect fluid, which has energy-momentum tensor

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p).$$

Using this energy-momentum tensor, and the FRW metric, one can derive the components of the Einstein equation, from which one gets the dynamical equations of the FRW universe:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

where $H = \dot{a}/a$. There is a final equation, which comes from energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$,

$$\dot{\rho} + 3H\left(\rho + p\right) = 0.$$

To "understand a cosmology" one derives relations between the scale factor a and energy density ρ as a function of time t. In doing so, it is standard practice to introduce an equation of state parameter w linking the energy density ρ and pressure p, via

$$p = w\rho.$$

Using this equation of state, the acceleration equation from the dynamical equations becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1 + 3w\right).$$

From this, one can notice (assuming $a, \rho > 0$) that if acceleration is to take place, $\ddot{a}(t) > 0$, then one requires $w < -\frac{1}{3}$. Now, as $w_{\mathrm{m}} = 0$ and $w_{\mathrm{r}} = \frac{1}{3}$ (the equations of state for non-relativistic matter and radiation, respectively), we clearly require some "other" component which has the correct equation of state. Infact, what we require, is that the universe currently has its total energy density dominated by this "other" component. Whilst the exact nature of this component is unknown, it is believed to exist, and is given the name *dark energy*.

II. REVIEW OF CURRENT DARK ENERGY MODELS

One of the current problems in modern cosmology is to provide a description of this dark energy component. In the literature there are many models, such as the vacuum energy associated with quantum field theory, quintessence, K-essence and elastic dark energy (among many others). In essence, three out of these four models rely on fields to provide the required dynamics and equation of state; hence, we shall review their generic properties.

The dynamics of a scalar field come from the extremisation of the action,

$$S = \int \mathrm{d}^4x \sqrt{-g}\,\mathcal{L}(\phi, \partial_\mu \phi)$$

which give the Euler-Lagrange equations of motion. The "canonical" Lagrangian describing a real scalar field $\phi(x^\mu)$, has the potential energy term subtracted from the kinetic term,

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\nu \phi \partial_\mu \phi - V(\phi), \tag{2}$$

where $V(\phi)$ is some potential the scalar field "lives in". The Euler-Lagrange equation of motion for the canonical Lagrangian (2) is

$$\partial^\mu \partial_\mu \phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0,$$

and its solution is that trajectory $\phi(x^\mu)$ which minimises the action. The energy-momentum tensor following this Lagrangian can be computed by varying the action with respect to the metric (which, in this case, is the Minkowski metric), as in (1), giving

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}. \tag{3}$$

One can read off the energy density and pressure of the scalar field, via the "00" and "ii" components of $T_{\mu\nu}$,

$$\rho_\phi = \frac{1}{2}\left(\dot{\phi}^2 - |\nabla \phi|^2\right) + V(\phi), \qquad p_\phi = \frac{1}{2}\left(\dot{\phi}^2 - |\nabla \phi|^2\right) - V(\phi).$$

Hence, using the definition of the equation of state $w = p/\rho$, this gives

$$w_\phi = \frac{\dot{\phi}^2 - |\nabla \phi|^2 - 2V(\phi)}{\dot{\phi}^2 - |\nabla \phi|^2 + 2V(\phi)}.$$

Immediately, we see that if the scalar field is homogeneous and static, (implying $\partial_i \phi = 0, \dot{\phi} = 0$, respectively) we simply have $w_\phi = -1$. In the study of slow-roll inflation, one approximates the kinetic energy component of a homogeneous field as being subdominant to the potential energy component (i.e. $\dot{\phi}^2 \ll V(\phi)$), so that the equation of state for the inflaton is $w_\phi = -1$, which is infact the equation of state required for exponential expansion. Until this assumption, the technologies behind inflation and a scalar field model of dark energy are identical. We now reinstate the time dependance of the scalar field.

A. Quintessence & K-essence

Quintessence models of dark energy refer to the class of models of homogeneous scalar fields, with canonical Lagrangian. In a scalar field dominated epoch (ie. the energy density contribution to the total ρ is dominated by ρ_ϕ, over matter and radiation), it can be shown that there exists solutions such that the equation of state fulfills the acceleration condition [5]. The solutions may start off with radiation domination, giving way to matter domination, but the resulting dominance can be tuned to be the scalar field; hence, at late time the universe accelerates. A useful feature of these models is the existence of attractor solutions (also termed "tracker solutions" in the literature) so that a wide variety of initial conditions give rise to the same late-time solution.

The equation of motion for such a quintessence scalar field (where we now use the FRW metric for an expanding universe) is easily derivable, and reads

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0,$$

with typical potentials being exponential $V(\phi) \propto e^{\kappa\phi}$ or power law $V(\phi) \propto \phi^\alpha$.

A very similar class of models are called K-essence models, whereby the usual kinetic term of the Lagrangian $\partial^\mu\phi\partial_\mu\phi$ is used as an argument of a function [6]. That is, the Lagrangian is composed of a function of the kinetic energy, with the potential subtracted in the usual way:

$$\mathcal{L}_K = K\left(g^{\mu\nu}\partial_\nu\phi\partial_\mu\phi\right) - V(\phi).$$

There are other models, with the scalar field "non-minimally coupled" to gravity [7]. In the above models, the interaction between ϕ and the metric $g_{\mu\nu}$ only entered via the integration measure $\sqrt{-g}$; such a model has a minimal coupling to gravity. The addition of a term to the Lagrangian, which is a coupled function, such as $\frac{1}{2}\phi^2 R$, would change the resulting Einstein equations. This effectively has the dark energy field coupled to matter fields.

B. The Cosmological Constant

The cosmological constant is one of the oldest models of dark energy, even though it did not start off as such. It started life as a term Einstein added to "fix" the universe to being static, then removed upon the discovery of the expansion of the universe. However, it has recently been reinserted as it provides a possible model of dark energy. Essentially, the cosmological constant is a homogeneous static scalar field.

Due to the symmetry properties of the metric tensor and Einstein's equation, one can add on an extra term without violating the general theory of relativity:

$$G_{\mu\nu} = \kappa T_{\mu\nu} + g_{\mu\nu}\Lambda.$$

As such an addition does not violate the covariance of the theory, one "should" add it – there is no *a priori* reason to neglect such a term. This can be modeled within the previously discussed framework as another species with equation of state $w_\Lambda = -1$. This component pervades the entire universe, equally and at all times: hence the term "constant". An entirely plausible theory for the origin of such a constant comes from the zero-point energy of a quantum harmonic oscillator. The idea however suffers from an horrific failure, as we shall demonstrate [8].

In non-relativistic quantum mechanics, the n^{th} mode of the harmonic oscillator has energy $E_n = \left(n + \frac{1}{2}\right)\omega$ – obviously in units where $\hbar = 1$. Hence, one can see that the ground state has energy $E_0 = \frac{1}{2}\omega$. The extension of the relativistic quantum theory, to the non-relativstic theory, is to use an infinite summation over modes to represent fields. This procedure always leaves a divergent integral over all ground state modes:

$$\rho_{\text{vac}} = \int_0^{k_{\max}} \frac{4\pi k^2 \mathrm{d}k}{2(2\pi)^3}\sqrt{\mathbf{k}^2 + m^2}.$$

In writing this expression we have been rather conservative in setting the upper limit to $k = k_{\max}$ rather than the proper $k = \infty$. Assuming that the quantum field theory, upon which this integral is based, holds to arbitrarily small length scales (which is the content of $k \to \infty$), is beyond the scope of the theory. That is, QFT is not designed to hold on arbitrarily small scales, where a quantum description of gravity is expected to become relevant. Hence, in this framework, we can only trust ρ_{vac} up to the Planck scale – that is, by imposing $k_{\max}$ to be a length scale down to which we trust QFT, we get the maximum vacuum energy we can trust. Roughly speaking, this gives a vacuum energy $\rho_{\text{vac}} \sim M_{\text{Pl}}^4 \sim (10^{18})^4\text{GeV}^4$. To put this value into context, using a merely microscopic cut-off value gives an energy density well above that required to close the universe. If one compares this value found by the zero point energy of QFT to the energy density of dark energy required to accelerate the universe, one finds

$$\rho_{\text{vac}}^{\text{accel}} \sim 10^{-120}\rho_{\text{vac}}.$$

This huge discrepancy has a number of consequences. First, the assumption about the mass scale appears incorrect, implying some other mass scale. Secondly, the cosmological constant does not therefore appear to come from the vacuum energy of QFT.

The frankly catastrophic difference in scales of the cosmological constant "solution" fuels the search for viable models of dark energy – something is out there in the universe making it accelerate, and we want to understand it. Related to this is the scales that must be chosen "by hand" in the quintessence models

III. THE ELASTIC DARK ENERGY MODEL

The class of models to which most of our efforts will be focused come from an idea that models dark energy as an elastic solid. Heuristically speaking, one can imagine that a grid of energy (the lines of the grid "are energy") pervades the universe, with the crucial property of the grid lines being such that their gravitational field is *repulsive*, and the equation of state is that required to accelerate the universe. To put a more formal skeleton over this statement, the model uses a frozen network of domain walls to provide a repulsive gravitational field, accelerating the universe. There are a number of immediate issues one must confront, upon making this conjecture: (a) how does a domain wall network form and (b) is the network stable? The general idea uses that the equation of state of a domain wall "gas" is $w_{\rm dw} = v^2 - \frac{2}{3}$ [9], so that for a static network, $v = 0$ and the resulting equation of state is that required to accelerate the universe. Such a network is called *frustrated.*

Almost all investigations into a large variety of domain wall models have been plagued by the scaling out of domain wall networks [10–14]. If domain walls scale out then they do not have the required equation of state, nulling the elastic dark energy hypothesis. Towards the end of our discussion we will discuss a set of models which do not scale out, and do provide a pseudo-stable domain wall network.

To give a complete account of the elastic dark energy model, one must discuss the technicalities underlying the model. In essence, topological defects are "things left over" after a phase transition changes the shape of a potential that some field lives in. Suppose a potential $V(\phi)$, at $T \gg T_{\rm c}$, where $T_{\rm c}$ is some critical temperature, has its minima invariant under the global actions of the symmetry group G. The space of minima of the potential is termed the vacuum manifold $\mathcal{M}$ of the theory. Then, as the potential T drops below $T_{\rm c}$, if $\mathcal{M}$ does not retain the invariance under G, then the symmetry of the vacuum manifold has been spontaneously broken; schematically denoted $G \to H$. In this notation, the vacuum manifold is given by the coset space $\mathcal{M} = G/H$. Classification of the resulting defects comes from the topology of the new vacuum manifold.

Without entering into the topological classifications, one can roughly categorise defects as follows. If the dimension of $\mathcal{M}$ is 0 (i.e. $\mathcal{M}$ is a collection of discrete points), then domain walls form, if $\dim(\mathcal{M}) = 1$ then cosmic strings are formed, if $\dim(\mathcal{M}) = 2$ monopoles and if $\dim(\mathcal{M}) = 3$ then textures. In all subsequent discussions we will restrict our attention to models for which domain walls are the resulting defect: the vacuum manifold is a collection of discrete points. A more formal definition would classify in terms of the dimension of a non-contractible curve within the manifold. For example, if a line is unshrinkable in $\mathcal{M}$, then cosmic strings are formed, if a surface is non-contractible then monopoles [15].

A. Spontaneous Symmetry Breaking

Here we discuss, and give example of, models which exhibit spontaneous symmetry breaking. The example we will come back to time and time again is the discrete Goldstone model, whereby a scalar field $\phi(x^\mu) \in \mathbb{R}$ lives in a potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2. \tag{4}$$

The model itself is invariant under (global) $\mathbb{Z}_2$ transformations of the field: $V(\phi) = V(-\phi)$; however, the vacuum manifold $\phi = \pm\eta$ does not share this invariance. To uncover this loss of invariance, we perturb the field about the vacuum manifold, $\phi = \eta + \psi$, and we find that the potential can be rewritten into

$$V(\psi) = \frac{\lambda}{4}\left(\psi^2 + 2\psi\eta\right)^2,$$

which is now obviously not $\mathbb{Z}_2$ invariant. The mass of the ψ in the vacuum manifold can be read off as $m_\psi^2 = 2\lambda\eta^2$. This broken symmetry upon occupation of the vacuum manifold is called spontaneous symmetry breaking. Another model possessing such a dynamic has a scalar field $\phi(x^\mu) \in \mathbb{C}$, described by the Abelian Higgs model,

$$\mathcal{L} = \left(\mathcal{D}_\mu\phi\right)^* \left(\mathcal{D}^\mu\phi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4}\left(|\phi|^2 - \eta^2\right)^2,$$

where the covariant derivative is $\mathcal{D}_\mu = \partial_\mu - \mathrm{i}eA_\mu$ and field strength tensor $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. This model is invariant under local (i.e. gauge) transformations,

$$\phi \longmapsto \phi' = e^{\mathrm{i}\theta(x^\mu)}\phi, \qquad A_\mu \longmapsto A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta.$$

Again, upon expanding the model about the vacuum manifold, $\mathcal{M} \cong S^1$, one finds non-zero masses for the particles with the $U(1)$ symmetry being lost, denoting spontaneous symmetry breaking. This scheme can be extended to non-Abelian models.

Basically, the feature which gives rise to topological defects is that when the field occupies the vacuum manifold of the theory, its value is non-zero. If the vacuum is a single point, then this non-zero value would give something like a cosmological constant; if the vacuum is composed of a set of degenerate points (whether connected or not), defects form.

An interesting concept of partial symmetry breaking comes when one consider breaking schemes [4]. Let us consider $SO(3)$. Then, if $\boldsymbol{\phi} \in SO(3)$, we can represent the field as a vector with real components,

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \boldsymbol{\phi} \in SO(3), \quad \phi_i \in \mathbb{R}.$$

Considering a Higgs potential of the form (4), the vacuum is obtained by the field configuration

$$\boldsymbol{\phi}^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = \eta^2.$$

Now, we are at liberty to choose how to construct this configuration, so let us choose the vacuum state to be

$$\boldsymbol{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}.$$

The interesting step comes when we consider the action of the generators of the group $SO(3)$ upon this vacuum state. The generators are

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, one can see that the only T_i that leaves the vacuum invariant is T_3 (infact, the vacuum state is annihilated, $T_3\boldsymbol{\phi}_0 = 0$). Therefore, upon choosing the vacuum state, the symmetry of the model has been reduced from $SO(3)$ down to $SO(2)$ (as it is an $SO(2)$ subspace $e^{\mathrm{i}\alpha T_3}$ which leaves the vacuum invariant). Schematically, this symmetry breaking scheme is denoted $SO(3) \to SO(2)$. Hence, the coset space is $\mathcal{M} = SO(3)/SO(2) \cong S^2$, the surface of a sphere, the topology of which permits the existence of monopoles.

B. Formation of Domain Walls

The standard prototypical model from which domain walls form arises when one considers a real scalar field in contact with a heat bath. We write the effective potential as having a contribution from the bare potential (i.e. the zero temperature potential) and from the Helmholtz free energy due to a heat bath at temperature T,

$$V_{\text{eff}}(\phi, T) = V(\phi) + f(T).$$

For a bosonic field the free energy can be written as

$$f(T) = -\frac{\pi^2}{90}T^4 + \frac{1}{24}m^2(\phi)T^2.$$

In field theory the mass $m^2(\phi)$ is defined as the coefficients of the quadratic terms of ϕ. Taking the mexican hat potential (4) as the bare potential, gives the effective potential as

$$V_{\text{eff}} = \frac{\lambda}{4}\phi^4 + \frac{1}{2}m_{\text{eff}}^2(T)\phi^2 + V_0,$$

where the effective mass is

$$m_{\text{eff}}^2(T) = \frac{\lambda}{4}\left(T^2 - 4\eta^2\right).$$

A trivial stability analysis reveals that for $T > 2\eta$ the only minimum of the potential is at the origin, and for $T < 2\eta$ there are now two degenerate minima not at the origin. Put more formally, the vacuum expectation value $\langle\phi\rangle = 0$ for $T > 2\eta$ and $\langle\phi\rangle = \pm\sqrt{-m_{\text{eff}}^2/\lambda}$ for $T < 2\eta$. This is an example of a bifurcation, whereby stability changes as a function of a model parameter. In all subsequent discussion we take the temperature to be zero, $T = 0$.

The mechanism by which domain walls themselves are formed is rather simple. Suppose there is a region of space at high temperature, with field in the vacuum manifold. Then, as the temperature drops below the critical temperature, the structure of the vacuum manifold changes – as the field evolves to minimise its energy, it must make a "decision" as to which of the minima to occupy. The field over all space does not necessarily choose the same minimum (with the maximum length scale over which the field occupies the same minima being ξ, the causal horizon distance). The spatial clumping of field in the same minima is a "domain" and the regions of space between such domains have the field continuously interpolating between the two, forming a "domain wall". It is this wall which is the defect – as the field interpolates between the minima it passes over the maximum, giving the domain wall a large amount of surface energy.

C. Properties of Domain Walls

To quantify the rather unusual statement of a "repulsive gravitational field", we will consider a specific model that allows domain walls. The classic model is the *discrete Goldstone model*, whereby a real scalar field $\phi(t, \mathbf{x})$ is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2,$$

where λ, η are real parameters of the model [15]. This potential has discrete minima at $\phi = \pm\eta$, and hence the solution to the equation of motion must be able to continuously interpolate between these two values. The static solution is

$$\phi(x) = \eta\tanh\left(\frac{x}{\Delta}\right), \qquad \Delta = \sqrt{\frac{2}{\lambda\eta^2}}, \tag{5}$$

which corresponds to a 1D kink along the x-axis – the solution is independent of the other two spatial directions. The steepness of the interpolation is directly proportional to Δ. Using the solution, one can compute the energy momentum tensor (3),

$$T^{\mu}_{\ \nu} = \frac{1}{2}\lambda\eta^4 \operatorname{sech}^4\left(\frac{x}{\Delta}\right) \operatorname{diag}(1,0,1,1).$$

One can note that the energy density will be maximum on the domain wall core, where $x = 0$. The tension of a wall is computed by integrating $T^0_{\ 0}$ over all space, giving $\sigma = \frac{4}{3}\sqrt{\lambda}\eta^3$. If the form of the energy-momentum tensor were $T^{\mu}_{\ \nu} \sim \operatorname{diag}(\rho, -p, -p, -p)$ then the Newtonian limit of Einstein's equation reads $\nabla^2\Phi = 4\pi G\,(\rho + 3p)$. Given the form of the domain wall energy-momentum tensor, the Newtonian limit of general relativity reads

$$\nabla^2\Phi = -4\pi G\rho.$$

It is this minus sign which denotes a repulsive gravitational field [15]. Related to this calculation, is the computation of the equation of state of a domain wall *gas*. Basically, one computes the equation of state for a space-averaged set of walls, compensating for walls moving at relativistic velocities. Using the form of the energy-momentum tensor, one can compute

$$w_{\rm dw} = v^2 - \frac{2}{3},$$

where v is the velocity of the gas [9].

There exists a conserved topological quantity associated with the domain wall "kink" [15]. If we consider the construction $J^{\mu} = \varepsilon^{\mu\nu}\partial_{\nu}\phi$, then the conservation equation $\partial_{\mu}J^{\mu} = 0$ is satisfied by tensor-symmetry considerations (as $\varepsilon_{\mu\nu}$ is the totally anti-symmetric Levi-Civita tensor, and partial derivatives commute). We can then expand the conservation equation, and integrate over all space. The integral over the term $\nabla\cdot\mathbf{J}$ goes to zero by exploiting the freedom to choose integration surface: at infinity the current $\mathbf{J}$ must not diverge. Thus, one is left with

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t} = 0, \qquad \mathcal{Q} \equiv \int \mathrm{d}^3x\, J_0.$$

Hence, considering a 1D kink, $J_0 = \partial_x\phi$, the topological charge $\mathcal{Q}$ is just

$$\mathcal{Q} = \phi(+\infty) - \phi(-\infty).$$

Therefore, if ϕ is in different vacua at $x = \pm\infty$, the kink is topologically stable.

D. The Model Zoo

There are a number of potentials which give rise to domain walls, as we now review. All models presented in this section have a vacuum manifold composed of a discrete set of points.

Vacuum manifolds are found by finding the field configurations that give the minimum of the potential.

As already mentioned, the simplest has a real scalar field $\phi(x^\mu)$ with potential given by

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 . \tag{6}$$

The most direct extension of this model is to a real vector field $\mathbf{\Phi}(x^\mu) = (\phi_1, \ldots, \phi_N)$, whereby a cubic anisotropy term breaks the otherwise global $O(N)$ symmetry. Such a model is given by the potential

$$V(\phi_i) = \frac{\lambda}{4}\left(|\mathbf{\Phi}|^2 - \eta^2\right)^2 + \epsilon \sum_{j=1}^{N} \phi_j^4, \tag{7}$$

and is called the cubic anisotropy model [16]. The vacuum manifold of this model is the set of vertices of an N-dim hypercube if $\epsilon > 0$ and the centres of the faces of that hypercube if $\epsilon < 0$.

A related model uses a complex field Φ in potential

$$V(\Phi) = \frac{\lambda}{4}\left(|\Phi|^2 - \eta^2\right)^2 + \epsilon|\Phi|^4 \cos(3\phi),$$

where $\Phi = |\Phi|e^{\mathrm{i}\phi}$. If $\epsilon = 0$ the vacuum manifold is the circle $|\Phi| = \eta$, and the system has a $U(1)$ symmetry. Breaking the symmetry with $\epsilon \neq 0$ picks out three equidistant points on that circle: the vacuum manifold is the set of 3 vertices of an equilateral triangle.

Carter's pentahedral model [17] uses two complex scalar fields Φ, Ψ, in the potential

$$V(\Phi, \Psi) \;=\; \frac{\lambda}{4}\left[\left(|\Phi|^2 - \eta^2\right)^2 + \left(|\Psi|^2 - \eta^2\right)^2\right] + \epsilon|\sigma|^2|\varphi|^2\left(\cos\theta + \cos\chi\right),$$

where $\theta = 2\phi + \psi, \chi = 2\psi - \phi$ and $\Phi = |\Phi|e^{\mathrm{i}\phi}, \Psi = |\Psi|e^{\mathrm{i}\psi}$. If $\epsilon = 0$ the model has a $U(1) \times U(1)$ symmetry; thus, the vacuum manifold with $\epsilon \neq 0$ picks out points on the surface of a torus, in the space of the field phases ϕ, ψ. Domains are formed in these field phases.

A rather more different (and complicated) model comes from the $SU(5) \times \mathbb{Z}_2$ model with $\mathcal{L} = \mathrm{Tr}(\partial_\mu \Phi) - V(\Phi)$ where $\Phi \in SU(5)$ [18, 19]. The model uses a potential

$$V(\Phi) = -m^2 \mathrm{Tr}(\Phi^2) + h\left(\mathrm{Tr}(\Phi^2)\right)^2 + \lambda \mathrm{Tr}(\Phi^4) + V_0, \tag{8}$$

and the field is expanded using generators of the $SU(5)$ Lie group,

$$\Phi(x^\mu) = f_1(x^\mu)\lambda_3 + f_2(x^\mu)\lambda_8 + f_3(x^\mu)\tau_3 + f_4(x^\mu)Y.$$

This model has an interesting feature as the (diagonal) topological charge matrix $\mathcal{Q}^{(i)}$ can be constructed in 5 different ways. The i^{th} matrix has unit entries except the ii^{th} entry which is -4. As the interaction between a kink and anti-kink, with charges $\mathcal{Q}^{(i)}$ and $\bar{\mathcal{Q}}^{(j)}$ respectively, is proportional to $\mathrm{Tr}\left(\mathcal{Q}^{(i)}\bar{\mathcal{Q}}^{(j)}\right)$, there are two possibilities for this number:

$$\mathrm{Tr}\left(\mathcal{Q}^{(i)}\bar{\mathcal{Q}}^{(j)}\right) = \begin{cases} -20 & i = j, \\ +5 & i \neq j. \end{cases}$$

The sign of the trace denotes if the interaction is attractive (negative) or repulsive (positive). Using this property, one can then construct kink lattices with nearest neighbour interactions being repulsive, by arranging the charge matrices in a particular way – thus stabilizing the lattice.

1. Conserved Continuous Symmetry

We now present a different type of model, whereby a field with broken discrete symmetry interacts with a second field with complete continuous symmetry. Schematically, the model has a $\mathbb{Z}_N \times U(1)$ symmetry. If a Lagrangian has field $\sigma(x^\mu) \in \mathbb{C}$, with a $U(1)$ symmetry, then Noether's theorem states that there exists a conserved 4-current,

$$J_\mu = \bar{\sigma}\partial_\mu\sigma - \sigma\partial_\mu\bar{\sigma},$$

and conserved charge

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0, \qquad Q = \int \mathrm{d}^3x\, J_0.$$

The motivation for this class of model is rather simple. If the conserved charge of the σ-field condenses on the domain walls, then this should provide a resisting force to the wall collapse – thus enabling domain walls to freeze into some configuration.

The first such model in the literature uses a Lagrangian describing kinky vortons [20], whereby the potential is written

$$V(\phi,\sigma) = \frac{\lambda_\phi}{4}\left(\phi^2 - \eta_\phi^2\right)^2 + \frac{\lambda_\sigma}{4}\left(|\sigma|^2 - \eta_\sigma^2\right)^2 + \beta\phi^2|\sigma|^2.$$

This model is obtainable from Witten's $U(1) \times U(1)$ cosmic vorton model, by restricting the string forming field to be real [21]. The corresponding equations of motion permit a solution with charged condensate "living" on the kink solution [22]. The present author has an extension of this model, whereby the cubic anisotropy model (7) interacts with the charged σ-field in an identical manner [23].

E. Model Evolution

Generally speaking, it is not difficult to create numerical simulations of a domain wall network, with computing power being the main restriction – generally simulations are done in $(2+1)$-dimensions rather than $(3+1)$. The usual methodology is rather simple: initially populate a grid with randomly assigned vacua, and let the system evolve by numerically solving the equations of motion.

The key quantity that analyses look for is the scaling dynamics of a domain wall network. If random initial conditions can be used to form domain walls, then the length of the set of domain walls – the network – can be computed. If the number of domain walls $N_{\rm dw}$ vary with time as $N_{\rm dw} \propto t^{-1}$, the network "scales out" – if one views the wall network at logarithmic time intervals, the number of walls halves as time progresses.

An analysis of the evolution discrete (6) and cubic anisotropy models (7) is presented in [10] (the discrete model has also been investigated in other works, such as [14]). Their general conclusion is that the number of walls varies with time as $N_{\rm dw} \propto t^{-1}$: the scaling solution. The $SU(5) \times \mathbb{Z}_2$ model (8) however has a different scaling dynamic, $N_{\rm dw} \propto t^{-\gamma}$, with $\gamma \approx 0.71$ [19]. The presence of the topological matrices provide the different evolution dynamic, as is somewhat to be expected. This different dynamic – almost to the point of network frustration, $\gamma \approx 0$ – is also observed in the kinky vorton model [20].

IV. THE OUTLOOK FOR ELASTIC DARK ENERGY

Whilst it is clear that some species is contributing to the overall energy density of the universe, such that its dominance accelerates the universe, the properties of the species are not clear. As we have presented in our review, there are many models which can reproduce the observed current acceleration, and to pin down which is the actual model, precise observational constraints are required. In the case of elastic dark energy, much work is still to be done in terms of model construction, so that resulting domain wall networks have the correct phenomenology (i.e. network frustration). Once such a model has been constructed, the job of producing observables remains – crudely put, the network should gravitationally lens, with a specific signature (which needs to be determined). Also, the symmetry breaking scale, characterised by η, should then be constrained.

Given our presentation above of the scaling dynamics, and the $N_{\rm dw} \propto t^{-\gamma}$, one requires a model which evolves to $\gamma \approx 0$, for wall frustration. So far, in the literature, this has only been hinted at by models with conserved charges (be they discrete or continuous).

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